

# New bounds and constructions for multiply constant-weight codes

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## Abstract

Multiply constant-weight codes (MCWCs) were introduced recently to improve the reliability of certain physically unclonable function response. In this paper, the bounds of MCWCs and the constructions of optimal MCWCs are studied. Firstly, we derive three different types of upper bounds which improve the Johnson-type bounds given by Chee *et al.* in some parameters. The asymptotic lower bound of MCWCs is also examined. Then we obtain the asymptotic existence of two classes of optimal MCWCs, which shows that the Johnson-type bounds for MCWCs with distances  $2\sum_{i=1}^m w_i - 2$  or  $2mw - w$  are asymptotically exact. Finally, we construct a class of optimal MCWCs with total weight four and distance six by establishing the connection between such MCWCs and a new kind of combinatorial structures. As a consequence, the maximum sizes of MCWCs with total weight less than or equal to four are determined almost completely.

## Index Terms

Multiply constant weight codes, spherical codes, Plotkin bound, Johnson bound, linear programming bound, Gilbert-Varshamov bound, concatenation, graph decompositions, skew almost-resolvable squares

## I. INTRODUCTION

Modern cryptographic practice rests on the use of one-way functions, which are easy to evaluate but difficult to invert. Unfortunately, commonly used one-way functions are either based on unproven conjectures or have known vulnerabilities. Physically unclonable functions (PUFs), introduced by Pappu *et al.* [20], provide innovative low-cost authentication methods and robust structures against physical attacks. Recently, PUFs have become a trend to provide security in low cost devices such as Radio Frequency Identifications (RFIDs) and smart cards [8], [14], [20], [23]. Multiply constant-weight codes (MCWCs) establish the connection between the design of the Loop PUFs [8] and coding theory, thus were put forward in [9]. In an MCWC, each codeword is a binary word of length  $mn$  which is partitioned into  $m$  equal parts and has weight exactly  $w$  in each part [9]. The more general definition of MCWCs with different lengths and weights in different parts can be found in [5]. This definition generalizes the classic definitions of constant-weight codes (CWCs) (where  $m = 1$ ) and doubly constant-weight codes (where  $m = 2$ ) [16], [19].

The theory of MCWCs is at a rudimentary stage. In [5] Chee *et al.* extended techniques of Johnson [16] and established certain preliminary upper and lower bounds for possible sizes of MCWCs. They also showed that these bounds are asymptotically tight up to a constant factor. In [7], Chee *et al.* gave some combinatorial constructions for MCWCs which yield several new infinite families of optimal MCWCs. In particular, by establishing the connection between MCWCs and combinatorial designs and using some existing results in design theory, they determined the maximum sizes of MCWCs with total weight less than or equal to four, leaving an infinite class open. In the same paper, they also showed that the Johnson-type bounds are asymptotically tight for fixed weights and distances by applying Kahn's Theorem [17] on the size of the matching in hypergraphs. Furthermore, in [6], they demonstrated that one of the Johnson-type bounds is asymptotically exact for the distance  $2mw - 2$ . This was achieved by applying the theory of edge-colored digraph-decompositions [18].

In this paper, we continue the study on the bounds of MCWCs and the constructions of optimal MCWCs. Our main contributions are as follows:

- We extend the techniques of Agrell *et al.* [4] and improve the Johnson-type bounds derived in [5]. We also show that the generalised Gilbert-Varshamov (GV) bound [15], [25] is better than the asymptotic lower bounds derived in [5], where the concatenation techniques are employed.
- We obtain the asymptotic existence of two classes of optimal MCWCs. One of them generalizes the known result of [6] for MCWCs with different weights in different parts. The other shows that another Johnson-type bound is asymptotically exact for distance  $2mw - w$ .

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- We consider the open case of optimal MCWCs in [7], i.e., doubly constant-weight codes with weight two in each part and distance six. We establish an equivalence relation between such MCWCs and certain kind of combinatorial structures, which are called skew almost-resolvable squares. Accordingly, several new constructions are proposed. As a consequence, the maximum sizes of MCWCs with total weight less than or equal to four are determined almost completely, leaving a very small number of lengths open.

The rest of this article is organized as follows. Section 2 collects the necessary definitions and notations. Section 3 gives three forms of upper bounds, which can improve the previous Johnson-type bounds. Section 4 studies the asymptotic lower bounds of MCWCs. Section 5 presents the asymptotic existence of two classes of optimal MCWCs. Section 6 handles the optimal MCWCs with total weight four. A conclusion is made in Section 7.

## II. DEFINITIONS AND NOTATIONS

### A. Multiply Constant-weight Codes

All sets considered in this paper are finite if not obviously infinite. We use  $[n]$  to denote the set  $\{1, 2, \dots, n\}$ . If  $X$  and  $R$  are finite sets,  $R^X$  denotes the set of vectors of length  $|X|$ . Each component of a vector  $\mathbf{u} \in R^X$  takes value in  $R$  and is indexed by an element of  $X$ , that is,  $\mathbf{u} = (\mathbf{u}_x)_{x \in X}$ , and  $\mathbf{u}_x \in R$  for each  $x \in X$ . A  $q$ -ary code of length  $n$  is a set  $\mathcal{C} \subseteq \mathbb{Z}_q^n$  for some  $X$  with size  $n$ . The elements of  $\mathcal{C}$  are called *codewords*. The *support* of a vector  $\mathbf{u} \in \mathbb{Z}_q^n$ , denoted  $\text{supp}(\mathbf{u})$ , is the set  $\{x \in X : \mathbf{u}_x \neq 0\}$ . The *Hamming norm* or the *Hamming weight* of a vector  $\mathbf{u} \in \mathbb{Z}_q^n$  is defined as  $\|\mathbf{u}\| = |\text{supp}(\mathbf{u})|$ . The distance induced by this norm is called the *Hamming distance*, denoted  $d_H$ , so that  $d_H(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$ , for  $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_q^n$ . A code  $\mathcal{C}$  is said to *have distance*  $d$  if the Hamming distance between any two distinct codewords of  $\mathcal{C}$  is at least  $d$ . A  $q$ -ary code of length  $n$  and distance  $d$  is called an  $(n, d)_q$  code. When  $q = 2$ , an  $(n, d)_2$  code is simply called an  $(n, d)$  code.

Let  $m, N$  be positive integers and  $X$  be a set of size  $N$ . Suppose that  $X$  can be partitioned as  $X = X_1 \cup X_2 \cup \dots \cup X_m$  with  $|X_i| = n_i$ ,  $i = 1, 2, \dots, m$ . An  $(N, d)$  code  $\mathcal{C} \subseteq \mathbb{Z}_2^N$  is said to be of *multiply constant-weight* and denoted by  $\text{MCWC}(w_1, n_1; w_2, n_2; \dots; w_m, n_m; d)$ , if each codeword has the weight  $w_1$  in the coordinates indexed by  $X_1$ , weight  $w_2$  in the coordinates indexed by  $X_2$ , and so on and so forth. When  $w_1 = w_2 = \dots = w_m = w$  and  $n_1 = n_2 = \dots = n_m = n$ , we simply denote this multiply constant-weight code of length  $N = mn$  by  $\text{MCWC}(m, n, d, w)$ .

The largest size of an  $(n, d)_q$  code is denoted by  $A_q(n, d)$ . When  $q = 2$ , the size is simply denoted by  $A(n, d)$ . The largest size of an  $\text{MCWC}(w_1, n_1; w_2, n_2; \dots; w_m, n_m; d)$  is denoted by  $T(w_1, n_1; w_2, n_2; \dots; w_m, n_m; d)$ ; the largest size of an  $\text{MCWC}(m, n, d, w)$  is denoted by  $M(m, n, d, w)$ ; and the largest size of a  $\text{CWC}(n, d, w)$  is denoted by  $A(n, d, w)$ . The code achieving the largest size is said to be *optimal*.

Next, we will restate the known results about MCWCs without proof, more details can be found in [5]. The authors of [5] first use the concatenation technique to construct MCWCs from the classic  $q$ -ary codes.

*Proposition 2.1:* ([5]) Let  $q \leq A(n, d_1, w)$ , we have

$$M(m, n, d_1 d_2, w) \geq A_q(m, d_2).$$

Specially,  $M(m, qw, 2d, w) \geq A_q(mw, d)$ .

As MCWC is a generalization of CWC, the techniques of Johnson for CWC [16] can be naturally extended to give the recursive bounds as follows:

*Proposition 2.2:* ([5])

$$T(w_1, n_1; w_2, n_2; \dots; w_m, n_m; d) \leq \lfloor \frac{n_i}{w_i} T(w_1, n_1; \dots; w_i - 1, n_i - 1; \dots; w_m, n_m; d) \rfloor, \quad (1)$$

$$T(w_1, n_1; w_2, n_2; \dots; w_m, n_m; d) \leq \lfloor \frac{n_i}{n_i - w_i} T(w_1, n_1; \dots; w_i, n_i - 1; \dots; w_m, n_m; d) \rfloor, \quad (2)$$

$$T(w_1, n_1; w_2, n_2; \dots; w_m, n_m; d) \leq \lfloor \frac{u}{w_1^2/n_1 + w_2^2/n_2 + \dots + w_m^2/n_m - \lambda} \rfloor, \quad (3)$$

where  $d = 2u$  and  $\lambda = w_1 + w_2 + \dots + w_m - u$ .

*Proposition 2.3:* ([5])

$$M(m, n, d, w) \leq \lfloor \frac{n^m}{w^m} M(m, n - 1, d, w - 1) \rfloor, \quad (4)$$

$$M(m, n, d, w) \leq \lfloor \frac{n^m}{(n - w)^m} M(m, n - 1, d, w) \rfloor, \quad (5)$$

$$M(m, n, d, w) \leq \lfloor \frac{d/2}{d/2 + mw^2/n - mw} \rfloor. \quad (6)$$

### B. Association Schemes

Let  $X$  be a finite set with at least two elements and, for any integer  $n \geq 1$ , let  $\mathcal{R} = \{R_0, R_1, \dots, R_n\}$  be a family of  $n+1$  relations  $R_i$  on  $X$ . The pair  $(X, \mathcal{R})$  will be called an *association scheme with  $n$  classes* if the following three conditions are satisfied:

1. The set  $\mathcal{R}$  is a partition of  $X^2$  and  $R_0$  is the diagonal relation, i.e.,  $R_0 = \{(x, x) | x \in X\}$ .
2. For  $i = 0, 1, \dots, n$ , the inverse  $R_i^{-1} = \{(y, x) | (x, y) \in R_i\}$  of the relation  $R_i$  also belongs to  $\mathcal{R}$ .
3. For any triple of integers  $i, j, k = 0, 1, \dots, n$ , there exists a number  $p_{i,j}^{(k)} = p_{j,i}^{(k)}$  such that, for all  $(x, y) \in R_k$ :

$$|\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}| = p_{i,j}^{(k)}. \quad (7)$$

The  $p_{i,j}^{(k)}$ 's are called the *intersection numbers* of the scheme  $(X, \mathcal{R})$ .

Any relation  $R_i$  can be described by its *adjacency matrix*  $D_i \in \mathbb{C}(X, X)$ , defined as follows:

$$D_i(x, y) = \begin{cases} 1, & (x, y) \in R_i, \\ 0, & (x, y) \notin R_i. \end{cases}$$

We call the linear space

$$A = \left\{ \sum_{i=0}^n \alpha_i D_i \mid \alpha_i \in \mathbb{C} \right\}$$

the *Bose-Mesner algebra* of the association scheme  $(X, \mathcal{R})$ . There is a set of pairwise orthogonal idempotent matrices  $J_0, J_1, \dots, J_n$ , which forms another basis of this Bose-Mesner algebra.

Given two bases  $\{D_k\}$  and  $\{J_k\}$  of the Bose-Mesner algebra of a scheme, let us consider the linear transformations from one into the other:

$$D_k = \sum_{i=0}^n P_k(i) J_i, \quad k = 0, 1, \dots, n.$$

From these we construct a square matrix  $P$  of order  $n+1$  whose  $(i, k)$ -entry is  $P_k(i)$ :

$$P = [P_k(i) : 0 \leq i, k \leq n].$$

Since  $P$  is nonsingular, there exists a unique square matrix  $Q$  of order  $n+1$  over  $\mathbb{C}$  such that

$$PQ = QP = |X|I.$$

The matrices  $P$  and  $Q$  are called the *eigenmatrices* of the association scheme.

Let  $\mathcal{R} = \{R_0, R_1, \dots, R_n\}$  be a set of  $n+1$  relations on  $X$  of an association scheme. For a nonempty subset  $Y$  of  $X$ , let us define the *inner distribution* of  $Y$  with respect to  $\mathcal{R}$  to be the  $(n+1)$ -tuple  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$  of nonnegative rational numbers  $\alpha_i$  given by

$$\alpha_i = |Y|^{-1} |R_i \cap Y^2|.$$

In [12], Delsarte gave a key observation about the inner distribution and the eigenmatrix  $Q$ .

*Theorem 2.4:* ([12]) The components  $\alpha Q_k$  of the row vector  $\alpha Q$  are nonnegative.

Let  $w$  and  $n$  be integers, with  $1 \leq w \leq n$ . In the Hamming space of dimension  $n$  over  $\mathbb{F} = \{0, 1\}$ , we consider the subset  $X$  of  $\mathbb{F}^n$  as follows:

$$X = \{x \in \mathbb{F}^n \mid w_H(x) = w\},$$

and we define the distance relations  $R_0, R_1, \dots, R_w$ :

$$R_i = \{(x, y) \in X^2 \mid d(x, y) = 2i\}.$$

For given  $n$  and  $w$ , with  $1 \leq w \leq n/2$ , we call  $(X, \mathcal{R})$  the *Johnson scheme*  $J(w, n)$ , i.e., binary codes with length  $n$  and constant weight  $w$ .

Given an integer  $k$ , with  $0 \leq k \leq w$ , we define *Eberlein polynomial*  $E_k(u)$ , in the indeterminate  $u$ , as follows:

$$E_k(u) = \sum_{i=0}^k (-1)^i \binom{u}{i} \binom{w-u}{k-i} \binom{n-w-u}{k-i}.$$

*Theorem 2.5:* ([12]) The eigenmatrices  $P$  and  $Q$  of the Johnson scheme  $J(w, n)$  are given by

$$P_k(i) = E_k(i),$$

$$Q_i(k) = \frac{\mu_i E_k(i)}{\binom{w}{i} \binom{n-w}{i}},$$

where  $\mu_i = \frac{n-2i+1}{n-i+1} \binom{n}{i}$ .

### C. Design Theory

To give our constructions of optimal MCWCs, we need the following notations and results in design theory.

Let  $K$  be a subset of positive integers and  $\lambda$  be a positive integer. A *pairwise balanced design*  $((v, K, \lambda)$ -PBD or  $(K, \lambda)$ -PBD of order  $v$ ) is a pair  $(X, \mathcal{B})$ , where  $X$  is a finite set (*the point set*) of cardinality  $v$  and  $\mathcal{B}$  is a family of subsets (*blocks*) of  $X$  that satisfy (1) if  $B \in \mathcal{B}$ , then  $|B| \in K$  and (2) every pair of distinct elements of  $X$  occurs in exactly  $\lambda$  blocks of  $\mathcal{B}$ . The integer  $\lambda$  is the *index* of the PBD. When  $K = \{k\}$ , a  $(v, \{k\}, \lambda)$ -PBD is also known as a *balanced incomplete block design* (BIBD), which is denoted by  $\text{BIBD}(v, k, \lambda)$ .

*Theorem 2.6 ([10]):* For any odd integer  $v \geq 5$ , a  $(v, \{5, 7, 9\}, 1)$ -PBD exists with exceptions  $v \in [11, 19] \cup \{23\} \cup [27, 33] \cup \{39\}$ , and possible exceptions  $v \in \{43, 51, 59, 71, 75, 83, 87, 95, 99, 107, 111, 113, 115, 119, 139, 179\}$ .

An  $\alpha$ -*parallel class* of blocks in a BIBD  $(X, \mathcal{B})$  is a subset  $\mathcal{B}' \subset \mathcal{B}$  such that each point  $x \in X$  is contained in exactly  $\alpha$  blocks in  $\mathcal{B}'$ . When  $\alpha = 1$ , we simply call it a *parallel class*, as usual. If the block set  $\mathcal{B}$  can be partitioned into  $\alpha$ -parallel classes, then the BIBD is called  $\alpha$ -*resolvable* (or just *resolvable* if  $\alpha = 1$ ). We will use  $\alpha$ -resolvable BIBDs to construct optimal MCWCs.

A *group divisible design* (GDD) is a triple  $(X, \mathcal{G}, \mathcal{B})$  where  $X$  is a set of points,  $\mathcal{G}$  is a partition of  $X$  into *groups*, and  $\mathcal{B}$  is a collection of subsets of  $X$  called *blocks* such that any pair of distinct points from  $X$  occurs either in some group or in exactly one block, but not both. A  $K$ -GDD of type  $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$  is a GDD in which every block has size from the set  $K$  and in which there are  $u_i$  groups of size  $g_i, i = 1, 2, \dots, s$ . When  $K = \{k\}$ , we simply write  $k$  for  $K$ . A  $k$ -GDD of type  $m^k$  is also called a *transversal design* and denoted by  $\text{TD}(k, m)$ .

*Theorem 2.7 ([1], [10]):* Let  $m$  be a positive integer. Then:

- 1) a  $\text{TD}(4, m)$  exists if  $m \notin \{2, 6\}$ ;
- 2) a  $\text{TD}(5, m)$  exists if  $m \notin \{2, 3, 6, 10\}$ ;
- 3) a  $\text{TD}(6, m)$  exists if  $m \notin \{2, 3, 4, 6, 10, 22\}$ ;
- 4) a  $\text{TD}(m+1, m)$  exists if  $m$  is a prime power.

### D. Decomposition of Edge-colored Complete Digraphs

Denote the set of all ordered pairs of a finite set  $X$  with distinct components by  $\overline{\binom{X}{2}}$ . An *edge-colored digraph* is a triple  $G = (V, C, E)$ , where  $V$  is a finite set of *vertices*,  $C$  is a finite set of *colors* and  $E$  is a subset of  $\overline{\binom{X}{2}} \times C$ . Members of  $E$  are called *edges*. The *complete edge-colored digraph* on  $n$  vertices with  $r$  colors, denoted by  $K_n^{(r)}$ , is the edge-colored digraph  $(V, C, E)$ , where  $|V| = n$ ,  $|C| = r$  and  $E = \overline{\binom{X}{2}} \times C$ .

A family  $\mathcal{F}$  of edge-colored subgraphs of an edge-colored digraph  $K$  is a *decomposition* of  $K$  if every edge of  $K$  belongs to exactly one member of  $\mathcal{F}$ . Given a family of edge-colored digraphs  $\mathcal{G}$ , a decomposition  $\mathcal{F}$  of  $K$  is a  $\mathcal{G}$ -*decomposition* of  $K$  if each edge-colored digraph in  $\mathcal{F}$  is isomorphic to some  $G \in \mathcal{G}$ . In [18], Lamken and Wilson exhibited the asymptotic existence of decompositions of  $K_n^{(r)}$  for a fixed family of digraphs. To state their result, we require more concepts.

Consider an edge-colored digraph  $G = (V, C, E)$  with  $|C| = r$ . Let  $((u, v), c) \in E$  denote a directed edge from  $u$  to  $v$ , colored by  $c$ . For any vertex  $u$  and color  $c$ , define the *indegree* and *outdegree* of  $u$  with respect to  $c$ , to be the number of directed edges of color  $c$  entering and leaving  $u$ , respectively. Then for vertex  $u$ , we define the *degree vector* of  $u$  in  $G$ , denoted by  $\tau(u, G)$ , to be the vector of length  $2r$ ,  $\tau(u, G) = (\text{in}_1(u, G), \text{out}_1(u, G), \dots, \text{in}_r(u, G), \text{out}_r(u, G))$ . Define  $\alpha(G)$  to be the greatest common divisor of the integers  $t$  such that the  $2r$ -vector  $(t, t, \dots, t)$  is a nonnegative integral linear combination of the degree vectors  $\tau(u, G)$  as  $u$  ranges over all vertices of all digraphs  $G \in \mathcal{G}$ .

For each  $G = (V, C, E) \in \mathcal{G}$ , let  $\mu(G)$  be the *edge vector* of length  $r$  given by  $\mu(G) = (m_1(G), m_2(G), \dots, m_r(G))$  where  $m_i(G)$  is the number of edges with color  $i$  in  $G$ . We denote by  $\beta(G)$  the greatest common divisor of the integers  $m$  such that  $(m, m, \dots, m)$  is a nonnegative integral linear combination of the vectors  $\mu(G)$ ,  $G \in \mathcal{G}$ . Then  $\mathcal{G}$  is said to be *admissible* if  $(1, 1, \dots, 1)$  can be expressed as a positive rational combination of the vectors  $\mu(G)$ ,  $G \in \mathcal{G}$ .

*Theorem 2.8 (Lamken and Wilson [18]):* Let  $\mathcal{G}$  be an admissible family of edge-colored digraphs with  $r$  colors. Then there exists a constant  $n_0 = n_0(\mathcal{G})$  such that a  $\mathcal{G}$ -decomposition of  $K_n^{(r)}$  exists for every  $n \geq n_0$  satisfying  $n(n-1) \equiv 0 \pmod{\beta(\mathcal{G})}$  and  $n-1 \pmod{\alpha(\mathcal{G})}$ .

In the same paper, the above theorem had also been extended to the multiplicity case. Consider the problem of finding a family  $\mathcal{F}$  of subgraphs of  $K_n^{(r)}$  each of which is isomorphic to a member of  $\mathcal{G}$ , so that each edge of  $K_n^{(r)}$  of color  $i$  occurs in exactly  $\lambda_i$  of the members of  $\mathcal{F}$ . We can think of this as a  $\mathcal{G}$ -decomposition of  $K_n^{[\lambda_1, \lambda_2, \dots, \lambda_r]}$ , which denotes the digraph on  $n$  vertices where there are exactly  $\lambda_i$  edges of color  $i$  joining  $x$  to  $y$  for any ordered pair  $(x, y)$  of distinct vertices.

Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  be a vector of positive integers. Let  $\alpha(\mathcal{G}; \lambda)$  denote the least positive integer  $t$  such that the constant vector  $t\lambda$  is an integral linear combination of  $\tau(u, G)$  as  $u$  ranges over all vertices of all digraphs  $G \in \mathcal{G}$ . Let  $\beta(\mathcal{G}; \lambda)$  denote the least positive integer  $m$  such that the constant vector  $m\lambda$  is an integral linear combination of  $\mu(G)$ ,  $G \in \mathcal{G}$ . We say  $\mathcal{G}$  is  $\lambda$ -*admissible* when the vector  $\lambda$  is a positive rational linear combination of  $\mu(G)$ ,  $G \in \mathcal{G}$ .

*Theorem 2.9 (Lamken and Wilson [18]):* Let  $\mathcal{G}$  be a  $\lambda$ -admissible family of edge- $r$ -colored digraphs, where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ . Then there exists a constant  $n_0 = n_0(\mathcal{G}, \lambda)$  such that a  $\mathcal{G}$ -decomposition of  $K_n^{[\lambda_1, \lambda_2, \dots, \lambda_r]}$  exists for every  $n \geq n_0$  satisfying  $n(n-1) \equiv 0 \pmod{\beta(\mathcal{G}; \lambda)}$  and  $n-1 \pmod{\alpha(\mathcal{G}; \lambda)}$ .

### III. UPPER BOUNDS

For the simplicity of illustration, when handling the general bounds of MCWCs, we only consider the special case of  $\text{MCWC}(m, n, d, w)$ . However, it is easy to see that our methods used can also be applied to the general case.

#### A. Bounds from Spherical Codes

We start with the definition of a spherical code. Different from the classic code, the spherical code is defined on the Euclidean space. A *spherical code* is a finite subset of  $S(n)$ , where  $S(n) := \{\mathbf{x} \in R^n : \|\mathbf{x}\| = 1\}$ . Here  $\|\cdot\|$  is the Euclidean norm. The *distance* between two codewords is defined by  $d_E(\mathbf{c}_1, \mathbf{c}_2) := \|\mathbf{c}_1 - \mathbf{c}_2\|$ . However, to characterize the codeword separation in a spherical code, the *minimum angle*  $\phi$  or the *maximum cosine*  $s$  is often used instead of the Euclidean distance. The relation between these three parameters is

$$s := \cos \phi = 1 - \frac{d_E^2}{2}.$$

We will generally use  $s$  as the separation parameter. The largest size of an  $n$ -dimensional spherical code with maximum cosine  $s$  is defined by  $A_S(n, s)$ .

When  $s \leq 0$ , the value of  $A_S(n, s)$  has been determined completely. [2], [11], [13], [21], [22]:

$$\begin{aligned} A_S(n, s) &= \lfloor 1 - \frac{1}{s} \rfloor, & \text{if } s \leq -\frac{1}{n}; \\ A_S(n, s) &= n + 1, & \text{if } -\frac{1}{n} \leq s < 0; \\ A_S(n, 0) &= 2n. \end{aligned}$$

Before proceeding further, let us remark that, under a suitable mapping, a binary code can be viewed as a spherical code. Thus an upper bound on the cardinality of the spherical code serves as an upper bound for the binary code. This observation can improve previous upper bounds in some cases.

Define

$$\mathcal{H}(n) = \{0, 1\}^n,$$

$$\mathcal{M}(m, n, w) = \{\mathbf{x} \in \mathcal{H}(n) : \mathbf{x} \cdot \mathbf{u}_i = w\},$$

where  $\mathbf{u}_i = \mathbf{e}_i \otimes \mathbf{j}_n$ ,  $\mathbf{e}_i$  is the standard  $m$ -dimensional unit vector and  $\mathbf{j}_n$  is the  $n$ -dimensional all-one vector. Then any subset of  $\mathcal{H}(n) = \{0, 1\}^n$  is a binary code of length  $n$  and any subset of  $\mathcal{M}(m, n, w)$  is an  $\text{MCWC}(m, n, d, w)$  for some distance  $d$ .

Let  $\Omega(\cdot)$  denote the mapping  $0 \rightarrow 1$  and  $1 \rightarrow -1$  from binary Hamming space to Euclidean space. Then

$$\Omega(\mathcal{M}(m, n, w)) = \{\mathbf{x} \in \Omega(\mathcal{H}(n)) : \mathbf{x} \cdot \mathbf{u}_i = n - 2w \text{ for } 1 \leq i \leq m\}.$$

For any point  $\mathbf{x} \in \mathcal{M}(m, n, w)$ ,  $\mathbf{x}$  satisfies  $(\Omega(\mathbf{x}) - \mathbf{x}_0) \cdot \mathbf{u}_i = 0$  and  $\|\Omega(\mathbf{x}) - \mathbf{x}_0\| = r$ , where

$$\mathbf{x}_0 = (1 - \frac{2w}{n})\mathbf{j}_{mn},$$

and

$$r = 2\sqrt{\frac{mw(n-w)}{n}}.$$

Hence  $\Omega(\mathcal{M}(m, n, w))$  is a subset of the  $(nm - m)$ -dimensional hypersphere of radius  $r$  centered at  $\mathbf{x}_0$ .

From the above analysis, we can get the following bound:

*Theorem 3.1:*

$$\begin{aligned} M(m, n, 2d, w) &\leq \lfloor \frac{d}{b} \rfloor, & \text{if } b \geq \frac{d}{nm-m+1}, \\ M(m, n, 2d, w) &\leq m(n-1) + 1, & \text{if } 0 < b < \frac{d}{n}, \end{aligned}$$

where

$$b = d - \frac{mw(n-w)}{n}.$$

*Proof:* Let  $\mathcal{C}$  be an  $\text{MCWC}(m, n, 2d, w)$ . Translating  $\Omega(\mathcal{C})$  by  $\mathbf{x}_0$  and scaling the radius by  $1/r$ , in accordance with the above analysis, yields an  $(nm - m)$ -dimensional spherical code with the maximum cosine  $s = 1 - \frac{dn}{mw(n-w)}$ . Thus

$$\begin{aligned} M(m, n, 2d, w) &\leq A_S(m(n-1), s), & \text{if } s \geq -1; \\ M(m, n, 2d, w) &= 1, & \text{if } s < -1. \end{aligned}$$

Using  $A_S(mn - m, s)$  as an upper bound for  $|\Omega(\mathcal{C})|$  completes the proof.  $\blacksquare$

*Remark 3.2:* The first bound in Theorem 3.1 is equivalent to the last Johnson-type bound (3) and the second bound improves the Johnson-type bound of Proposition 2.2 when  $0 < b < \frac{d}{n}$ .



### B. Plotkin-type Bounds

The following proposition is well-known, while we provide a sketch of the proof for the sake of completeness.

*Proposition 3.3:* ([4]) Let  $\mathcal{C}$  be an  $(n, d)$  code, then

$$|\mathcal{C}| \leq \frac{d/2}{d/2 - \sum_{i=1}^n f_i(1 - f_i)}$$

provided that the denominator is positive, where  $f_i$  denotes the proportion of codewords that have a 1 in position  $i$ .

*Proof:* The proof follows from the technique of double counting. On one hand,

$$d_{av} = \frac{1}{M(M-1)} \sum_{c_1, c_2 \in \mathcal{C}} d(c_1, c_2) \geq d,$$

where  $M = |\mathcal{C}|$ . On the other hand,

$$d_{av} = \frac{2M}{M-1} \sum_{i=1}^n f_i(1 - f_i).$$

By the double counting principle,

$$\frac{2M}{M-1} \sum_{i=1}^n f_i(1 - f_i) \geq d.$$

For MCWCs, we will have more restrictions concerning  $f_i$ , so we expect to get a better bound. ■

*Theorem 3.4:*

$$M(m, n, 2d, w) \leq \max \left\{ \frac{d}{d - \sum_{i=1}^{mn} f_i(1 - f_i)} \right\} \quad (8)$$

where the maximum is taken over all  $f_i$  ( $1 \leq i \leq mn$ ) that satisfy the constraints below:

$$f_1 + f_2 + \cdots + f_n = w \quad (9)$$

$$f_{n+1} + f_{n+2} + \cdots + f_{2n} = w \quad (10)$$

$$\vdots \quad (11)$$

$$f_{(m-1)n+1} + f_{(m-1)n+2} + \cdots + f_{mn} = w. \quad (12)$$

*Proof:* The proof follows from the definition of MCWCs and Proposition 3.3. ■

*Corollary 3.5:*

$$M(m, n, 2d, w) \leq \lfloor \frac{d}{b} \rfloor, \quad (13)$$

where

$$b = d - \frac{mw(n-w)}{n}.$$

*Proof:* To get an upper bound of MCWCs, we only need to determine the minimum value of  $\sum_{i=1}^n f_i^2$ , when  $f_1 + f_2 + \cdots + f_n = w$ . We use the method of Lagrange Multiplier. Let  $\gamma$  be an auxiliary variable. We consider the following function:

$$g(f_1, f_2, \dots, f_n, \gamma) = \sum_{i=1}^n f_i^2 + \gamma(f_1 + f_2 + \cdots + f_n - w).$$

Then

$$\frac{\partial g}{\partial f_i} = 2f_i + \gamma = 0,$$

$$\frac{\partial g}{\partial \gamma} = \sum_{i=1}^n f_i - w = 0.$$

Thus when  $f_i = \frac{w}{n}$ , the original function will achieve the minimum value. Substituting  $f_i$  with  $\frac{w}{n}$  in the sum of (8), we obtain (13). ■

*Remark 3.6:* The bound (13) is equivalent to the Johnson-type bound (6) of Proposition 2.3, however when we impose the additional constraint that  $f_i$  must be multiples of  $1/M$ , the problem will be set in the discrete domain  $\{0, 1/M, 2/M, \dots, 1\}$  instead of the continuous domain  $[0, 1]$ . Similar with the above discussion of Corollary 3.5, we will get an implicit expression of the upper bound.

*Corollary 3.7:* If  $b > 0$ , then

$$M(m, n, 2d, w) \leq \lfloor d/b \rfloor,$$

where

$$\begin{aligned} b &= d - \frac{mw(n-w)}{n} + \frac{nm}{M^2} \{Mw/n\} \{M(n-w)/n\}, \\ M &= M(m, n, 2d, w), \\ \{x\} &= x - \lfloor x \rfloor. \end{aligned}$$

### C. Linear Programming Bounds

Let  $\mathcal{C}$  be an MCWC( $m, n, 2d, w$ ). The distance distribution of  $\mathcal{C}$  can be defined as follows:

$$A_{2i_1, 2i_2, \dots, 2i_m} := \frac{1}{|\mathcal{C}|} \sum_{\mathbf{c} \in \mathcal{C}} A_{2i_1, 2i_2, \dots, 2i_m}(\mathbf{c}),$$

where  $A_{2i_1, 2i_2, \dots, 2i_m}(\mathbf{c}) := |\{\mathbf{c}_1 \in \mathcal{C} : (\mathbf{c}_1 \oplus \mathbf{c}) \cdot \mathbf{u}_j = 2i_j\}|$ ,  $\mathbf{u}_j := \mathbf{e}_j \otimes \mathbf{j}_n$ ,  $\mathbf{e}_j$  is the standard  $m$ -dimensional unit vector and  $\mathbf{j}_n$  is the  $n$ -dimensional all-one vector.

*Corollary 3.8:* Let  $\mathcal{C}$  be an MCWC( $m, n, 2d, w$ ), then

$$\sum_{i_1=0}^w \sum_{i_2=0}^w \cdots \sum_{i_m=0}^w Q_{k_1}(i_1) Q_{k_2}(i_2) \cdots Q_{k_m}(i_m) A_{2i_1, 2i_2, \dots, 2i_m} \geq 0.$$

*Proof:* For  $v = 1, 2, \dots, m$ , suppose  $(X^{(v)}; R_0^{(v)}, \dots, R_w^{(v)})$  is an association scheme with intersection numbers  $p_{ijk}^{(v)}$ , incidence matrices  $D_i^{(v)}$ , idempotents  $J_i^{(v)}$ , and eigenvalues  $P_k^{(v)}(i)$ ,  $Q_k^{(v)}(i)$ . Then the Cartesian product  $(X^{(1)} \times X^{(2)} \times \cdots \times X^{(m)}; R_{i_1 \dots i_m} = R_{i_1}^{(1)} \times \cdots \times R_{i_m}^{(m)}, 0 \leq i_j \leq m \text{ for } 1 \leq j \leq m)$  is an association scheme with eigenmatrices  $Q_{k_1}^{(1)}(i_1) Q_{k_2}^{(2)}(i_2) \cdots Q_{k_m}^{(m)}(i_m)$ . Hence  $\mathcal{C}$  is a code in the product of  $m$  Johnson schemes. The result follows from Theorem 2.4.  $\blacksquare$

*Theorem 3.9:*

$$M(m, n, 2d, w) \leq 1 + \lfloor \max \sum_{i_1=0}^w \sum_{i_2=0}^w \cdots \sum_{i_m=0}^w A_{2i_1, \dots, 2i_m} \rfloor,$$

where

$$\begin{aligned} A_{2i_1, \dots, 2i_m} &\geq 0, \\ A_{2i_1, \dots, 2i_m} &= 0, \quad \text{for } \sum_{j=1}^m i_j < d; \end{aligned}$$

and

$$\sum_{i_1=0}^w \sum_{i_2=0}^w \cdots \sum_{i_m=0}^w Q_{k_1}(i_1) Q_{k_2}(i_2) \cdots Q_{k_m}(i_m) A_{2i_1, 2i_2, \dots, 2i_m} \geq 0. \quad (14)$$

## IV. ASYMPTOTIC LOWER BOUNDS

In this section, we consider the asymptotic rate of  $M(m, n, d, w)$  when  $m$  is large,  $n$  is a function of  $m$ ,  $d = \lfloor \delta mn \rfloor$  and  $w = \lfloor \omega n \rfloor$  for  $0 < \delta, \omega < 1$ . Define the value  $\mu(\delta, \omega)$  as follows:

$$\mu(\delta, \omega) := \limsup_{m \rightarrow \infty} \frac{\log_2 M(m, n, \lfloor \delta mn \rfloor, \lfloor \omega n \rfloor)}{mn}.$$

In [5], Chee *et al.* used the concatenation technique to give the following asymptotic lower bound.

*Proposition 4.1:* ([5]) For  $\delta \leq 1/2$ , we have

$$\mu(\delta, 1/2) \geq 1 - H(\delta),$$

where  $H(x)$  denotes the binary entropy function defined by

$$H(x) := -x \log_2 x - (1-x) \log_2 (1-x),$$

for all  $0 \leq x \leq 1$ .

In this section, we will generalise Proposition 4.1 and give a general form of the asymptotic lower bound. After that, we will give a generalised Gilbert-Varshamov bound for MCWCs and show that this classic method can provide a better bound.

The first bound follows from Proposition 2.1. We choose the  $q$ -ary code that can achieve the Gilbert-Varshamov bound as outer codes. For convenience, we assume  $\frac{1}{\omega}$  and  $\delta mn$  are integers.

*Theorem 4.2:* For  $\omega \leq 1/2$  and  $\delta \leq \max\{1/2, 2\omega\}$ , we have

$$\mu_c(\delta, \omega) \geq \omega \log_2 \left( \frac{1}{\omega} \right) \left( 1 - H_{\frac{1}{\omega}} \left( \frac{\delta}{2\omega} \right) \right),$$

where  $H_q(x) := x \log_q(q-1) - x \log_q x - (1-x) \log_q(1-x)$  for  $0 < x \leq \frac{q-1}{q}$ .

*Proof:* Applying Proposition 2.1, we get  $M(m, n, \delta mn, \omega n) \geq A_{\frac{1}{\omega}}(mwn, \frac{\delta mn}{2})$ . Since  $A_q(n, d) \geq q^{(1-H_q(d/n))n}$ , then

$$M(m, n, \delta mn, \omega n) \geq \left(\frac{1}{\omega}\right)^{(1-H_{\frac{1}{\omega}}(\frac{\delta}{2\omega}))mwn},$$

thus

$$\mu_c(\delta, \omega) \geq \omega \log_2\left(\frac{1}{\omega}\right)(1 - H_{\frac{1}{\omega}}(\frac{\delta}{2\omega})).$$

*Remark 4.3:* Actually, there exist algebraic geometric codes leading to an asymptotic improvement upon Gilbert-Varshamov bound when the alphabet size  $q \geq 49$  [24], [26]. Since the improvement is slight, we still use the Gilbert-Varshamov bound for the sake of simplicity.

The Gilbert-Varshamov bound is one of the most well-known and fundamental results in coding theory. In fact, it can be easily applied to various kinds of codes. For MCWC( $m, n, 2d, w$ ), the volume of the Hamming ball of radius  $2d-1$  is

$$\sum_{i_1+i_2+\dots+i_m \leq d-1} \binom{w}{i_1} \binom{n-w}{i_1} \cdots \binom{w}{i_m} \binom{n-w}{i_m}.$$

*Theorem 4.4:* For  $\omega \leq 1/2$  and  $\delta \leq \max\{1/2, 2\omega\}$ , we have

$$\mu_{GV}(\delta, \omega) \geq H_2(\omega) - \omega H_2\left(\frac{\delta}{2\omega}\right) - (1-\omega) H_2\left(\frac{\delta}{2(1-\omega)}\right).$$

*Proof:* Since

$$\begin{aligned} M(m, n, \delta mn, \omega n) &\geq \frac{\binom{n}{\omega n}^m}{\sum_{i_1+i_2+\dots+i_m \leq \frac{\delta mn}{2}-1} \binom{\omega n}{i_1} \binom{(1-\omega)n}{i_1} \cdots \binom{\omega n}{i_m} \binom{(1-\omega)n}{i_m}} \\ &\geq \frac{\binom{n}{\omega n}^m}{\sum_{0 \leq i \leq \frac{\delta mn}{2}} \binom{\omega mn}{i} \binom{(1-\omega)mn}{i}}, \end{aligned}$$

we have

$$\begin{aligned} \mu_{GV}(\delta, \omega) &\geq \frac{\log_2 \frac{2^{nmH_2(\omega)}}{2^{\omega nm H_2(\frac{\delta}{2\omega})} 2^{(1-\omega)mn H_2(\frac{\delta}{2(1-\omega)})}}}{mn} \\ &\geq H_2(\omega) - \omega H_2\left(\frac{\delta}{2\omega}\right) - (1-\omega) H_2\left(\frac{\delta}{2(1-\omega)}\right). \end{aligned}$$

At the end of this section, we compare the two bounds given above and show that the generalised Gilbert-Varshamov bound offers a better one.

*Theorem 4.5:*

$$\mu_{GV}(\delta, \omega) \geq \mu_c(\delta, \omega),$$

equality holds only when  $w = \frac{1}{2}$  or  $\delta = 2(\omega - \omega^2)$ .

*Proof:* Let

$$\begin{aligned} f(\delta, \omega) &= \mu_{GV}(\delta, \omega) - \mu_c(\delta, \omega) \\ &= H_2(\omega) - (1-\omega) H_2\left(\frac{\delta}{2(1-\omega)}\right) + \frac{\delta}{2} \log_2\left(\frac{1}{\omega} - 1\right) - (1-\omega) \log_2(1-\omega). \end{aligned}$$

For simplicity, letting  $x = \frac{\delta}{2}$ , we get

$$f(x, \omega) = -(2-2\omega-x) \log_2(1-\omega) + x \log_2\left(\frac{x}{\omega}\right) + (1-\omega-x) \log_2(1-\omega-x).$$

We will derive the proof by considering two cases of  $\omega \leq \frac{1}{4}, x \leq \omega$  and  $\frac{1}{4} < \omega \leq \frac{1}{2}, x \leq \frac{1}{4}$  separately.

(a)  $\omega \leq \frac{1}{4}, x \leq \omega$ .

When  $x = 0$ ,  $f(0, \omega) = -(1-\omega) \log_2(1-\omega) > 0$ .

When  $x = \omega$ ,  $f(\omega, \omega) = (3\omega-2) \log_2(1-\omega) - (2\omega-1) \log_2(1-2\omega)$ . We want to show  $f(\omega, \omega) \geq 0$ . Since  $f(0, 0) = 0$  and  $f(\frac{1}{4}, \frac{1}{4}) = 2 - \frac{5}{4} \log_2 3 > 0$ , we need to show that  $g(\omega) = f(\omega, \omega)$  is monotonely increasing.

$$g'(\omega) = 3 \log_2(1-\omega) - 2 \log_2(1-2\omega) + \frac{\omega}{\omega-1},$$

$$g''(\omega) = \frac{\omega(3-2\omega)}{(\omega-1)^2(1-2\omega)} > 0.$$



Since  $g'(0) = 0$  and  $g'(\frac{1}{4}) = \frac{5}{3} + 3\log_2(\frac{3}{4}) > 0$ , we get  $g'(\omega) \geq 0$ , thus  $f(\omega, \omega) \geq 0$ .

Moreover  $\frac{\partial f(x, \omega)}{\partial x} = \log_2 \frac{x(1-\omega)}{\omega(1-\omega-x)} = 0$ , we get  $x = \omega - \omega^2$ . Since  $f(\omega - \omega^2, \omega) = 0$ , with the above analysis, we get  $f(\delta, \omega) \geq 0$ .

(b)  $\frac{1}{4} < \omega \leq \frac{1}{2}, x \leq \frac{1}{4}$ .

When  $x = 0$ ,  $f(0, \omega) = -(1 - \omega) \log_2(1 - \omega) > 0$ .

When  $x = \frac{1}{4}$ ,  $f(\frac{1}{4}, \omega) = -(\frac{7}{4} - 2\omega) \log_2(1 - \omega) + \frac{1}{4} \log_2(\frac{1}{4\omega}) + (\frac{3}{4} - \omega) \log_2(\frac{3}{4} - \omega)$ . We want to show  $f(\frac{1}{4}, \omega) \geq 0$ . Since  $f(\frac{1}{4}, \frac{1}{4}) = -\frac{5}{4} \log_2(\frac{3}{4}) - \frac{1}{2} > 0$  and  $f(\frac{1}{4}, \frac{1}{2}) = 0$ , we show the function  $f(\frac{1}{4}, \omega)$  is monotonely decreasing.

$$f'(\frac{1}{4}, \omega) = \frac{1}{\ln 2} (2 \ln(1 - \omega) - \ln(\frac{3}{4} - \omega) + 1 - \frac{1}{4(1 - \omega)} - \frac{1}{4\omega}),$$

$$f''(\frac{1}{4}, \omega) = \frac{1}{\ln 2} (\frac{1 - 2\omega}{4\omega^2(1 - \omega)^2}) \geq 0.$$

Since  $f'(\frac{1}{4}, \frac{1}{4}) = \frac{1}{\ln 2} (\ln(\frac{9}{8}) - \frac{1}{3}) < 0$  and  $f'(\frac{1}{4}, \frac{1}{2}) = 0$ , we get  $f'(\frac{1}{4}, \omega) \leq 0$ , thus  $f(\frac{1}{4}, \omega) \geq 0$ .

The remainder of the proof is the same as the first case. Then, we have already proven this theorem. ■

## V. TWO INFINITE CLASSES OF OPTIMAL CODES

In [6], Chee *et al.* demonstrated that certain Johnson-type bounds are asymptotically exact for constant-composition codes, nonbinary constant-weight codes and MCWCs by constructing several infinite classes of optimal codes achieving these bounds. Especially, for MCWCs they showed that the bound (1) is asymptotically exact for distance  $2mw - 2$ .

*Theorem 5.1 (Chee et al. [6]):* Fix  $m$  and  $w$ . There exists an integer  $n_0$  such that

$$M(m, n, 2mw - 2, w) = \frac{n(n-1)}{w^2}$$

for all  $n \geq n_0$  satisfying  $n - 1 \equiv 0 \pmod{w^2}$ .

In this section, we will generalize Theorem 5.1 to the case where the weight  $w_i$  may not be equal. We determine the value of  $T(w_1, n; w_2, n; \dots; w_m, n; 2 \sum_{i=1}^m w_i - 2)$  for some modulo classes of  $n$  when  $n$  is sufficiently large. We also establish the connection between  $\alpha$ -resolvable BIBDs and MCWCs and employ Theorem 2.9 to establish the asymptotic existence of a class of  $\alpha$ -resolvable BIBDs. As a consequence, we prove that the bound (3) is asymptotically exact for distance  $2mw - w$ .

### A. Optimal MCWCs with Distance $2 \sum_{i=1}^m w_i - 2$

Let  $w_1 \geq w_2 \geq \dots \geq w_m$  be nonnegative integers. Let  $w = \sum_{i=1}^m w_i$ . The Johnson-type bound (1) shows that

$$T(w_1, n; w_2, n; \dots; w_m, n; 2w - 2) \leq \begin{cases} \frac{n(n-1)}{w_1(w_1-1)}, & \text{if } w_1 > w_2; \\ \frac{n(n-1)}{w_1^2}, & \text{if } w_1 = w_2. \end{cases}$$

We will show that this bound is asymptotically tight. To apply Theorem 2.8, we first define the family of edge-colored digraphs  $\mathcal{G}$ . We use the  $m^2$  ordered pairs from  $[m]$  as colors. Define  $\overline{w} = [w_1, w_2, \dots, w_m]$ . Let  $G(\overline{w})$  be the digraph with vertex set

$$V(G(\overline{w})) = W_1 \cup W_2 \cup \dots \cup W_m \tag{15}$$

where  $W_i$ 's are disjoint vertex sets with  $|W_i| = w_i$ . Here, for all distinct  $x, y \in V(G(\overline{w}))$ , there is an edge from  $x$  to  $y$  of color  $(i, j)$  where  $i$  and  $j$  are such that  $x \in W_i$  and  $y \in W_j$ . Then in the graph  $G(\overline{w})$ , there are  $w_i w_j$  edges colored  $(i, j)$  with  $i \neq j$ , and  $w_i(w_i - 1)$  edges colored  $(i, i)$ . For  $i, j \in [m]$ , let  $G_{ij}$  be a digraph with two vertices and one directed edge of color  $(i, j)$ . To define  $\mathcal{G}(\overline{w})$ , we consider the following two cases depending on whether  $w_1 = w_2$ :

- 1) When  $w_1 > w_2$ , we have  $w_1(w_1 - 1) \geq w_1 w_2$ . Let  $r$  be the largest integer such that  $w_1 - 1 = w_2 = \dots = w_r$ . Then set  $\mathcal{G}(\overline{w}) = \{G(\overline{w})\} \cup \{G_{ij} : (i, j) \in ([m] \times [m]) \setminus \{(1, i), (i, 1) : 1 \leq i \leq r\}\}$ .
- 2) When  $w_1 = w_2$ , we have  $w_1 w_2 > w_1(w_1 - 1)$ . Let  $r$  be the largest integer such that  $w_1 = \dots = w_r$ . Then set  $\mathcal{G}(\overline{w}) = \{G(\overline{w})\} \cup \{G_{ij} : (i, j) \in ([m] \times [m]) \setminus \binom{[r]}{2}\}$ .

*Proposition 5.2:* Suppose that a  $G(\overline{w})$ -decomposition of  $K_n^{(m^2)}$  exists. Then

$$T(w_1, n; \dots; w_m, n; 2w - 2) = \begin{cases} \frac{n(n-1)}{w_1(w_1-1)}, & \text{if } w_1 > w_2; \\ \frac{n(n-1)}{w_1^2}, & \text{if } w_1 = w_2. \end{cases}$$

*Proof:* Let  $V$  be the vertex set of  $K_n^{(m^2)}$  and  $\mathcal{F}$  be the  $G(\overline{w})$ -decomposition. Let  $X = \{1, 2, \dots, m\} \times V$ . The code is constructed in  $2^X$ . For each  $F \in \mathcal{F}$  isomorphic to  $G(\overline{w})$ , there is a unique partition of the vertex set  $V(F) = \cup_{i=1}^m S_i$  so that the edge from  $x$  to  $y$  in  $F$  has color  $(i, j)$  if  $x \in S_i$  and  $y \in S_j$ . Construct a codeword  $\mathbf{u}$  such that  $\mathbf{u}_{(i,x)} = 1$  if  $x \in S_i$ , and

$\mathbf{u}_{(i,x)} = 0$  otherwise. Since  $|S_i| = w_i$ , this code is an MCWC( $w_1, n; \dots; w_m, n; d$ ) with some distance  $d$ . Noting that every colored edge appears at most once in the member of  $\mathcal{F}$  isomorphic to  $G(\overline{w})$ , we have  $|\text{supp}(\mathbf{u}) \cap \text{supp}(\mathbf{v})| \leq 1$  for any two codewords  $\mathbf{u}$  and  $\mathbf{v}$ . Thus this code has distance  $2w - 2$ .

Finally, let  $m$  be the number of digraphs in  $\mathcal{F}$  isomorphic to  $G(\overline{w})$ . It is easy to see that  $m = \frac{n(n-1)}{w_1(w_1-1)}$  if  $w_1 > w_2$  and  $m = \frac{n(n-1)}{w_1^2}$  otherwise. ■

Noting that  $m_{(i,j)}(G(\overline{w})) = w_i w_j$ ,  $i \neq j$ ,  $m_{(i,i)}(G(\overline{w})) = w_i(w_i - 1)$  and  $m_{(i,j)}(G_{ij}) = 1$ , we have

$$\beta(\mathcal{G}(\overline{w})) = \begin{cases} w_1(w_1 - 1), & \text{if } w_1 > w_2; \\ w_1^2, & \text{if } w_1 = w_2. \end{cases}$$

Since  $\text{in}_{(i,j)}(G(\overline{w})) = w_j$ ,  $\text{out}_{(i,j)}(G(\overline{w})) = w_i$  for any  $i \neq j$ ,  $\text{in}_{(i,i)}(G(\overline{w})) = \text{out}_{(i,i)}(G(\overline{w})) = w_i - 1$ , it is easy to check that

$$\alpha(\mathcal{G}(\overline{w})) = \begin{cases} w_1(w_1 - 1), & \text{if } w_1 > w_2; \\ w_1, & \text{if } w_1 = w_2. \end{cases}$$

Then applying Theorem 2.8, we can obtain the following result.

*Theorem 5.3:* Let  $w_1 \geq w_2 \geq \dots \geq w_m$  be nonnegative integers and  $w = \sum_{i=1}^m w_i$ . There exists an integer  $n_0$  such that

$$T(w_1, n; \dots; w_m, n; 2w - 2) = \begin{cases} \frac{n(n-1)}{w_1(w_1-1)}, & \text{if } w_1 > w_2; \\ \frac{n(n-1)}{w_1^2}, & \text{if } w_1 = w_2. \end{cases}$$

for all  $n \geq n_0$  satisfying  $n - 1 \equiv 0 \pmod{w_1(w_1 - 1)}$  if  $w_1 > w_2$ , or  $n - 1 \equiv 0 \pmod{w_1^2}$  otherwise.

### B. Optimal MCWCs with Distance $2mw - w$

We first establish a connection between  $\alpha$ -resolvable BIBDs and optimal MCWCs.

*Proposition 5.4:* If there exists an  $\alpha$ -resolvable BIBD( $v, k, \lambda$ ), then  $M(m, n, d, w) = v$ , where  $m = \frac{\lambda(v-1)}{\alpha(k-1)}$ ,  $n = \frac{\alpha v}{k}$ ,  $d = 2(\frac{\lambda(v-1)}{k-1} - \lambda)$ , and  $w = \alpha$ .

*Proof:* The Johnson-type bound (3) shows that  $M(m, n, d, w) \leq v$  where  $m = \frac{\lambda(v-1)}{\alpha(k-1)}$ ,  $n = \frac{\alpha v}{k}$ ,  $d = 2(\frac{\lambda(v-1)}{k-1} - \lambda)$ , and  $w = \alpha$ .

Let  $(X, \mathcal{B})$  be an  $\alpha$ -resolvable BIBD( $v, k, \lambda$ ). Since there are  $\frac{\lambda(v-1)}{\alpha(k-1)}$   $\alpha$ -parallel classes in  $\mathcal{B}$ , each of which consists of  $\frac{\alpha v}{k}$  blocks, we can arrange all the blocks in an  $m \times n$  array with  $m = \frac{\lambda(v-1)}{\alpha(k-1)}$  and  $n = \frac{\alpha v}{k}$ , such that the blocks in each row form an  $\alpha$ -parallel class. Now, for each point  $x \in X$ , construct a codeword  $\mathbf{u}$  with  $\mathbf{u}_{(i,j)} = 1$  if the block in the entry  $(i, j)$  contains  $x$ , and  $\mathbf{u}_{(i,j)} = 0$  otherwise. Since each point appears in  $\alpha$  times in each row, the code constructed above is an MCWC( $m, n, d, \alpha$ ) of size  $v$  for some distance  $d$ . Since any two distinct points of  $X$  appear together in exactly  $\lambda$  blocks, the supports of any two codewords intersect in exactly  $\lambda$  points. Thus the code has distance  $d = 2(mw - \lambda) = 2(\frac{\lambda(v-1)}{k-1} - \lambda)$ . ■

In the remaining of this subsection, we employ Theorem 2.9 to show that when  $\alpha = \lambda$  and  $k \mid \alpha$ , an  $\alpha$ -resolvable BIBD( $v, k, \lambda$ ) exists for all sufficient  $v$  with  $v \equiv 1 \pmod{k-1}$ .

We first define the family of edge- $r$ -colored digraphs  $\mathcal{G}$  with  $r = k^2 - k$ . We use the  $(k-1)^2$  ordered pairs from  $[k-1]$  and the  $k-1$  singletons  $(i)$ ,  $i = 1, 2, \dots, k-1$  as colors. Let  $\lambda$  be a vector of length  $k^2 - k$  with each entry being  $\lambda$ . For each  $(k-1)$ -tuple  $\mathbf{t} = (t_1, t_2, \dots, t_{k-1})$  of nonnegative integers summing to  $k$ , let  $G(\mathbf{t})$  be the digraph with  $k+1$  vertices

$$V(G(\mathbf{t})) = \{w\} \cup T_1 \cup T_2 \cup \dots \cup T_{k-1} \quad (16)$$

where  $T_i$ 's are disjoint vertex sets with  $|T_i| = t_i$  and  $w$  is another vertex not in any  $T_i$ . Here, for all distinct  $x, y \in V(G(\mathbf{t}))$ , there is an edge from  $x$  to  $y$  of color  $(i, j)$  where  $i$  and  $j$  are such that  $x \in T_i$  and  $y \in T_j$ , and an edge of color  $(i)$  from the special vertex  $w$  to each  $x$  in  $T_i$ . Let  $\mathcal{G}$  be the collection of all such  $G(\mathbf{t})$ .

*Proposition 5.5:* If there exists a  $\mathcal{G}$ -decomposition of the edge- $r$ -colored  $K_m^{[\lambda, \lambda, \dots, \lambda]}$  with  $r = k^2 - k$  and  $m = \frac{v-1}{k-1}$ , then a  $\lambda$ -resolvable BIBD( $m(k-1) + 1, k, \lambda$ ) exists.

*Proof:* Let  $V$  be the vertex set of  $K_m^{[\lambda, \lambda, \dots, \lambda]}$  and let  $X = \{\infty\} \cup (V \times [k-1])$ . Let  $B_x = \{\infty\} \cup (\{x\} \times \{1, 2, \dots, k-1\})$ ,  $\mathcal{B} = \{B_x : x \in V\}$ . The elements  $V$  will be used to index the  $\lambda$ -parallel classes, which are denoted as  $\mathcal{P}_x$ ,  $x \in V$ ;  $B_x$  will be in  $\mathcal{P}_x$ . For each  $F \in \mathcal{F}$ , there will be a unique partition of the  $k+1$  vertices  $V(F) \subset V$  as

$$V(F) = \{w\} \cup S_1 \cup S_2 \cup \dots \cup S_{k-1}$$

as in (16). Let

$$A_F = \bigcup_{i=1}^{k-1} S_i \times \{i\};$$

we take  $\lambda$  copies of this block in the parallel class  $\mathcal{P}_w$ . Let  $\mathcal{A} = \{A_F : F \in \mathcal{F}\}$  and let  $\mathcal{B}^\lambda$  be a multi-set containing each member of  $\mathcal{B}$   $\lambda$  times. It is easy to check that  $(X, \mathcal{A} \cup \mathcal{B}^\lambda)$  is a  $((k-1)m + 1, k, \lambda)$ -BIBD, and that each  $\mathcal{P}_w$  is a  $\lambda$ -parallel

class. For example, the  $\lambda$  blocks in  $\mathcal{P}_w$  that contains a point  $(y, i)$ ,  $y \neq w$  are  $A_F$ 's where  $F$ 's are the graphs in  $\mathcal{F}$  that contain the edge of color  $(i)$  from  $w$  to  $y$ . ■

With the same argument as that in the proof of [18, Theorem 10.1], one can show that  $m(m-1)(\lambda, \lambda, \dots, \lambda)$  is an integral linear combination of the vectors  $\mu(G(\mathbf{t}))$ ,  $G(\mathbf{t}) \in \mathcal{G}$ , and  $(m-1)(\lambda, \lambda, \dots, \lambda)$  is an integral linear combination of the vectors  $\tau(x, G(\mathbf{t}))$  as  $x$  ranges over all vertices of all digraphs  $G(\mathbf{t}) \in \mathcal{G}$ . Thus, the two conditions of Theorem 2.9 are satisfied. Applying this theorem we can obtain the following result.

**Theorem 5.6:** Given positive integers  $k$  and  $\lambda$  with  $k \mid \lambda$ , there exists a constant  $m_0 = m_0(k, \lambda)$  such that a  $\lambda$ -resolvable BIBD( $m(k-1) + 1, k, \lambda$ ) exists for all  $m \geq m_0$ .

Combining Proposition 5.4 and Theorem 5.6, we can get the following result.

**Theorem 5.7:** Given positive integers  $k$  and  $w$  with  $k \mid w$ , there exists a constant  $m_0 = m_0(k, w)$  such that

$$M(m, n, 2(mw - w), w) = m(k - 1) + 1$$

with  $n = w(m(k-1) + 1)/k$  for all  $m \geq m_0$ .

## VI. OPTIMAL MCWCs WITH WEIGHT FOUR

In [7], the authors determined the maximum size of MCWCs for total weight less than or equal to four, except when  $m = 2$ ,  $w_1 = w_2 = 2$ ,  $d = 6$  and  $n_1 \leq n_2 \leq 2n_1 - 1$ , with both  $n_1$  and  $n_2$  being odd. We consider this open class in this section. The Johnson-type bound (1) yields that:

**Lemma 6.1:** Let  $n_1, n_2$  be two odd integers with  $0 < n_1 \leq n_2 \leq 2n_1 - 1$ . Then  $T(2, n_1; 2, n_2; 6) \leq \lfloor \frac{n_2(n_1-1)}{4} \rfloor$ .

We will show the above bound can be achieved for most cases. Firstly, we introduce a new combinatorial structure and establish the connection between such a structure and the optimal MCWC( $2, n_1; 2, n_2; 6$ ).

### A. Skew Almost-resolvable Squares

Let  $V$  be a set of  $v$  points and  $S$  be a set of  $s$  points. A *skew almost-resolvable square*, denoted  $\text{SAS}(s, v)$ , is an  $s \times s$  array, where the rows and the columns are indexed by the elements of  $S$ , and each cell is either empty or contains a pair of points from  $V$ , such that:

- 1) for every two cells  $(i, j)$  and  $(j, i)$  with  $i \neq j$  at most one is filled;
- 2) the cells on the diagonal are all empty;
- 3) no pair of points from  $V$  appears in more than one cell;
- 4) for each  $i \in S$ , the pairs in row  $i$  together with those in column  $i$  form a partition of  $V \setminus \{x\}$  for some  $x \in V$ .

**Proposition 6.2:** Let  $v \equiv 1 \pmod{4}$  and  $s \equiv 1 \pmod{2}$  with  $v \leq s \leq 2v - 1$ . There exists an MCWC( $2, v; 2, s; 6$ ) of size  $\frac{s(v-1)}{4}$  if and only if an  $\text{SAS}(s, v)$  exists.

*Proof:* Let  $A$  be an  $\text{SAS}(s, v)$  on  $V$  with rows and columns indexed by  $S$ . We may assume that  $V$  and  $S$  are distinct. Let  $X = V \cup S$ . The code is constructed in  $2^X$ . For each filled cell  $(i, j)$  of  $A$  with  $A(i, j) = \{a, b\}$ , construct a codeword  $\mathbf{u}$  where  $\mathbf{u}_x = 1$  if  $x \in \{a, b, i, j\}$ , and  $\mathbf{u}_x = 0$  otherwise. Then we get an MCWC( $2, v; 2, s; d$ ) for some distance  $d$ . Note that Properties 1), 3) and 4) guarantee that any pair of points of  $X$  appear in at most one codeword's support. The supports of any two distinct codewords  $\mathbf{u}$  and  $\mathbf{v}$  intersect in at most one point and then the code has distance 6. According to Property 4), for each  $i \in S$ , there are  $\frac{v-1}{2}$  cells filled in row  $i$  and column  $i$ . Thus we have  $\frac{s(v-1)}{4}$  cells filled in total and the code has size  $\frac{s(v-1)}{4}$ .

Conversely, let  $X = X_1 \cup X_2$  with  $|X_1| = v$  and  $|X_2| = s$ . Let  $\mathcal{C}$  be an MCWC( $2, v; 2, s; 6$ ) of size  $\frac{s(v-1)}{4}$  in  $2^X$ . Construct an  $s \times s$  array with rows and columns indexed by the elements of  $S$ . For each codeword  $\mathbf{u} \in \mathcal{C}$  with  $\text{supp}(\mathbf{u}) = \{a, b, i, j\}$ ,  $a, b \in X_1$  and  $i, j \in X_2$ , fill in the cell  $(i, j)$  with the pair  $\{a, b\}$ . It is easy to check that this array is an  $\text{SAS}(s, v)$ . ■

In the above definition of SASs, if we replace the condition 4) by the following one, we get the definition of  $\text{SAS}^*(s, v)$ .

- 4)' there exists an  $i_0 \in S$  such that for each  $i \in S \setminus \{i_0\}$ , the pairs in row  $i$  and column  $i$  form a partition of  $V \setminus \{x\}$  for some  $x \in V$ ; the pairs in row  $i_0$  and column  $i_0$  form a partition of  $V \setminus \{x, y, z\}$  for some distinct  $x, y, z \in V$ .

Similarly, we have the following result, the proof of which is exactly the same as that of Proposition 6.2 and we omit it here.

**Proposition 6.3:** Let  $v \equiv 3 \pmod{4}$  and  $s \equiv 1 \pmod{2}$  with  $v \leq s \leq 2v - 1$ . There exists an MCWC( $2, v; 2, s; 6$ ) of size  $\lfloor \frac{s(v-1)}{4} \rfloor$  if and only if an  $\text{SAS}^*(s, v)$  exists.

In the following, we will discuss a useful construction method, i.e., frame construction, which will allow us to construct infinite families of SASs and  $\text{SAS}^*$ s.

Let  $V$  be a set of  $v$  points and  $S$  be a set of  $s$  points. Let  $\{H_1, H_2, \dots, H_n\}$  be a partition of  $V$  with  $|H_i| = h_i$  and  $\{S_1, S_2, \dots, S_n\}$  be a partition of  $S$  with  $|S_i| = s_i$ . A *skew frame-resolvable square* (SFS) of type  $\{(s_i, h_i) : 1 \leq i \leq n\}$  is an  $s \times s$  array, where the rows and the columns are indexed by the elements of  $S$ , and each cell is either empty or contains a pair of points from  $V$ , such that:

- 1) for every two cells  $(i, j)$  and  $(j, i)$  with  $i \neq j$  at most one is filled;

- 2) the subarray indexed by  $S_i \times S_i$  is empty, and it is called *hole*;
- 3) no pair of points from  $V$  appears in more than one cell;
- 4) no pair of points from  $H_i$  appears in any cell;
- 5) for each  $l \in S_i$ , the pairs in row  $l$  together with those in column  $l$  form a partition of  $V \setminus H_i$ .

We will use an exponential notation  $(s_1, g_1)^{n_1} \cdots (s_n, g_n)^{n_n}$  to indicate that there are  $n_i$  occurrences of  $(s_i, g_i)$  in the partitions.

We can use GDDs to give the recursive construction of SFSs.

**Construction 6.4:** Let  $(X, \mathcal{G}, \mathcal{B})$  be a GDD, and let  $s, v : X \rightarrow \mathbb{Z}^+ \cup \{0\}$  be two weight functions on  $X$ . Suppose that for each block  $B \in \mathcal{B}$ , there exists an SFS of type  $\{(s(x), v(x)) : x \in B\}$ . Then there is an SFS of type  $\{(\sum_{x \in G} s(x), \sum_{x \in G} v(x)) : G \in \mathcal{G}\}$ .

*Proof:* For each  $x \in X$ , let  $S(x)$  be an index set of  $s(x)$  elements, where  $S(x)$  and  $S(y)$  are disjoint for any  $x \neq y \in X$ . For each  $B \in \mathcal{B}$ , we construct an SFS of type  $\{(s(x), v(x)) : x \in B\}$   $\mathcal{A}_B$  on  $\cup_{x \in B} (\{x\} \times \{1, 2, \dots, v(x)\})$  and index its rows and columns using the elements of the set  $\cup_{x \in B} S(x)$ .

Denote  $S = \cup_{x \in X} S(x)$  and  $V = \cup_{x \in X} (\{x\} \times \{1, 2, \dots, v(x)\})$ . We construct the requisite SFS  $\mathcal{A}$  on  $V$  and index its rows and columns by  $S$  as follows: for each cell of  $\mathcal{A}$  indexed by  $(\alpha, \beta)$ , if  $\alpha \in S(x)$ ,  $\beta \in S(y)$  with  $x \neq y$  and there exists a block  $B \in \mathcal{B}$  containing  $x, y$ , then we place the entry from  $\mathcal{A}_B$  indexed by  $(\alpha, \beta)$  in the cell of  $\mathcal{A}$ ; otherwise the cell is empty.

For each  $G_i \in \mathcal{G}$ , denote  $S_i = \cup_{x \in G_i} S(x)$  and  $H_i = \cup_{x \in G_i} (\{x\} \times \{1, 2, \dots, v(x)\})$ . It is easy to check that Properties 1) – 4) in the definition of SFSs are satisfied. Now, for each  $\alpha \in S_i$ , we consider the pairs in row  $\alpha$  and column  $\alpha$ . Assume that  $\alpha \in S(x)$  for some  $x \in G_i$ . Since for each  $y \notin G_i$ , there exists a unique block containing both  $x$  and  $y$ , the set  $\{B \setminus \{x\} : x \in B \in \mathcal{B}\}$  forms a partition of  $X \setminus G_i$ . Note that for each  $\mathcal{A}_B$  with  $x \in B$ , the pairs in row  $\alpha$  and column  $\alpha$  of  $\mathcal{A}_B$  form a partition of  $\cup_{y \in B, y \neq x} (y \times \{1, 2, \dots, v(y)\})$ . Then the pairs in row  $\alpha$  and column  $\alpha$  in  $\mathcal{A}$  form a partition of

$$\bigcup_{x \in B, B \in \mathcal{B}} \left( \bigcup_{y \in B, y \neq x} (y \times \{1, 2, \dots, v(y)\}) \right) = \bigcup_{y \in X \setminus G_i} (y \times \{1, 2, \dots, v(y)\}) = V \setminus H_i.$$

Thus we have proved that  $\mathcal{A}$  is an SFS of type  $\{(\sum_{x \in G} s(x), \sum_{x \in G} v(x)) : G \in \mathcal{G}\}$ .  $\blacksquare$

Let  $V$  be a set of  $v$  points and  $S$  be a set of  $s$  points. Let  $W$  be a subset of  $V$  with  $|W| = w$  and  $T$  be a subset of  $S$  with  $|T| = t$ . A *holey skew almost-resolvable square*, denoted  $\text{HSAS}(s, v; t, w)$ , is an  $s \times s$  array, where the rows and the columns are indexed by the elements of  $S$ , and each cell is either empty or contains a pair of points from  $V$ , such that:

- 1) for every two cells  $(i, j)$  and  $(j, i)$  with  $i \neq j$  at most one is filled;
- 2) the subarray indexed by  $T \times T$  is empty, and it is called *hole*;
- 3) no pair of points from  $V$  appears in more than one cell;
- 4) no pair of points from  $W$  appears in any cell;
- 5) for each  $t \in T$ , the pairs in row  $t$  together with those in column  $t$  form a partition of  $V \setminus W$ ;
- 6) for each  $l \in S \setminus T$ , the pairs in row  $l$  and column  $l$  form a partition of  $V \setminus \{x\}$  for some  $x \in V$ .

The following result is simple but useful in our constructions.

**Proposition 6.5:** Suppose that there exist both an  $\text{HSAS}(s, v; t, w)$  and an  $\text{SAS}(t, w)$ . Then an  $\text{SAS}(s, v)$  exists.

In the following, we show how to construct SASs from SFSs.

**Construction 6.6:** [Basic Frame Construction] Suppose that there exists an SFS of type  $\{(s_i, h_i) : 1 \leq i \leq n\}$ . Let  $s = \sum_{i=1}^n s_i$  and  $v = \sum_{i=1}^n h_i$ . If for each  $1 \leq i \leq n-1$  there exists an  $\text{HSAS}(s_i + e, h_i + w; e, w)$ , furthermore,

- (1) if there exists an  $\text{HSAS}(s_n + e, h_n + w; e, w)$ , then an  $\text{HSAS}(s + e, v + w; e, w)$  exists;
- (2) if there exists an  $\text{SAS}(s_n + e, h_n + w)$ , then an  $\text{SAS}(s + e, v + w)$  exists;
- (3) if there exists an  $\text{SAS}^*(s_n + e, h_n + w)$ , then an  $\text{SAS}^*(s + e, v + w)$  exists.

*Proof:* Let  $\mathcal{A}$  be an SFS of type  $\{(s_i, h_i) : 1 \leq i \leq n\}$  on  $V = \cup_{i=1}^n H_i$  with rows and columns indexed by  $S$ . Let  $W$  be a set of size  $w$ , disjoint from  $V$ , and take our new point set to be  $V \cup W$ . Now, add  $e$  new rows and columns. For each  $1 \leq i \leq n-1$ , fill the  $s_i \times s_i$  subsquare together with the  $e$  new rows and columns with a copy of the  $\text{HSAS}(s_i + e, h_i + w; e, w)$  on  $H_i \cup W$ , such that the intersection of the new rows and columns forms a hole. Then, fill the  $s_n \times s_n$  subsquare together with the  $e$  new rows and columns with a copy of the  $\text{HSAS}(s_n + e, h_n + w; e, w)$  ( $\text{SAS}(s_n + e, h_n + w; e, w)$ ,  $\text{SAS}^*(s_n + e, h_n + w; e, w)$ ). It is routine to check that the resultant square is an  $\text{HSAS}(s + e, v + w; e, w)$  ( $\text{SAS}(s + e, v + w)$ ,  $\text{SAS}^*(s + e, v + w)$ ).  $\blacksquare$

## B. Determining the Value of $T(2, n_1; 2, n_2; 6)$

Suppose  $\mathbf{u} \in \mathbb{Z}_2^X$  is a codeword of an MCWC( $w_1, n_1; w_2, n_2; d$ ). We can represent  $\mathbf{u}$  equivalently as a 4-tuple  $\langle a_1, a_2, a_3, a_4 \rangle \in X^4$ , where  $\mathbf{u}_{a_1} = \mathbf{u}_{a_2} = \mathbf{u}_{a_3} = \mathbf{u}_{a_4} = 1$ . Throughout this section, we shall often represent codewords of MCWCs in this form.

**Lemma 6.7:** Let  $n_1 \in \{3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 29, 33, 37\}$ ,  $n_1 \leq n_2 \leq 2n_1 - 1$  and  $n_2$  be odd. Then

- 1)  $T(2, n_1; 2, n_2; 6) = \lfloor \frac{n_2(n_1-1)}{4} \rfloor$ , except for  $(n_1, n_2) = (5, 7)$ ; furthermore
- 2)  $T(2, 5; 2, 7; 6) = 6$ .

TABLE I  
CODEWORDS OF SMALL MCWC( $2, n_1; 2, n_2; 6$ ) FOR  $3 \leq n_1 \leq 9$

$(n_1, n_2)$	Codewords									
(3, 3)	$\langle 0, 1, 3, 4 \rangle$									
(3, 5)	$\langle 0, 1, 3, 4 \rangle \langle 1, 2, 5, 6 \rangle$									
(5, 5)	$\langle 0, 1, 5, 6 \rangle \langle 0, 2, 7, 8 \rangle \langle 1, 3, 7, 9 \rangle \langle 2, 4, 5, 9 \rangle \langle 3, 4, 6, 8 \rangle$									
(5, 7)	$\langle 0, 1, 5, 6 \rangle \langle 0, 2, 7, 8 \rangle \langle 0, 3, 9, 10 \rangle \langle 1, 2, 9, 11 \rangle \langle 1, 4, 7, 10 \rangle \langle 3, 4, 5, 8 \rangle$									
(5, 9)	$\langle 0, 3, 10, 9 \rangle \langle 2, 3, 5, 13 \rangle \langle 0, 2, 8, 7 \rangle \langle 0, 4, 11, 12 \rangle \langle 1, 2, 9, 11 \rangle \langle 1, 3, 7, 12 \rangle \langle 0, 1, 6, 5 \rangle \langle 1, 4, 8, 13 \rangle \langle 2, 4, 6, 10 \rangle$									
(7, 7)	$\langle 0, 1, 7, 8 \rangle \langle 0, 2, 9, 10 \rangle \langle 0, 3, 11, 12 \rangle \langle 1, 2, 11, 13 \rangle \langle 1, 4, 9, 12 \rangle \langle 2, 5, 7, 12 \rangle \langle 3, 4, 7, 10 \rangle \langle 3, 5, 8, 9 \rangle \langle 4, 6, 8, 11 \rangle \langle 5, 6, 10, 13 \rangle$									
(7, 9)	$\langle 0, 1, 7, 8 \rangle \langle 0, 2, 9, 10 \rangle \langle 0, 3, 11, 12 \rangle \langle 0, 4, 13, 14 \rangle \langle 1, 2, 11, 13 \rangle \langle 1, 3, 9, 14 \rangle \langle 1, 4, 10, 12 \rangle \langle 2, 3, 7, 15 \rangle \langle 2, 5, 8, 12 \rangle$ $\langle 3, 5, 10, 13 \rangle \langle 4, 5, 7, 9 \rangle \langle 4, 6, 8, 11 \rangle \langle 5, 6, 14, 15 \rangle$									
(7, 11)	$\langle 0, 1, 7, 8 \rangle \langle 0, 2, 9, 10 \rangle \langle 1, 5, 11, 13 \rangle \langle 0, 4, 13, 14 \rangle \langle 0, 5, 15, 16 \rangle \langle 3, 6, 16, 13 \rangle \langle 1, 3, 9, 14 \rangle \langle 0, 3, 11, 17 \rangle \langle 1, 6, 15, 17 \rangle$ $\langle 2, 3, 7, 15 \rangle \langle 2, 4, 8, 16 \rangle \langle 2, 5, 12, 14 \rangle \langle 3, 5, 8, 10 \rangle \langle 1, 4, 12, 10 \rangle \langle 4, 5, 7, 17 \rangle \langle 4, 6, 9, 11 \rangle$									
(7, 13)	$\langle 0, 1, 7, 8 \rangle \langle 0, 2, 9, 10 \rangle \langle 0, 3, 11, 12 \rangle \langle 1, 4, 12, 10 \rangle \langle 5, 4, 7, 9 \rangle \langle 0, 6, 17, 18 \rangle \langle 1, 2, 11, 13 \rangle \langle 1, 3, 9, 14 \rangle \langle 3, 5, 10, 8 \rangle$ $\langle 1, 5, 17, 19 \rangle \langle 3, 4, 19, 18 \rangle \langle 2, 4, 8, 16 \rangle \langle 5, 0, 16, 15 \rangle \langle 0, 4, 14, 13 \rangle \langle 2, 3, 7, 17 \rangle \langle 3, 6, 13, 16 \rangle \langle 4, 6, 11, 15 \rangle \langle 2, 5, 12, 18 \rangle$ $\langle 2, 6, 14, 19 \rangle$									
(9, 9)	$\langle 7, 3, 15, 12 \rangle \langle 2, 1, 16, 11 \rangle \langle 4, 8, 9, 15 \rangle \langle 0, 3, 11, 10 \rangle \langle 2, 8, 10, 13 \rangle \langle 2, 6, 9, 12 \rangle \langle 4, 5, 11, 12 \rangle \langle 1, 0, 12, 17 \rangle \langle 6, 7, 13, 17 \rangle$ $\langle 6, 0, 14, 15 \rangle \langle 6, 4, 10, 16 \rangle \langle 1, 4, 14, 13 \rangle \langle 7, 1, 10, 9 \rangle \langle 7, 8, 14, 11 \rangle \langle 3, 8, 17, 16 \rangle \langle 5, 2, 15, 17 \rangle \langle 3, 5, 14, 9 \rangle \langle 0, 5, 13, 16 \rangle$									
(9, 11)	$\langle 4, 8, 13, 11 \rangle \langle 3, 0, 14, 10 \rangle \langle 6, 5, 11, 19 \rangle \langle 3, 1, 16, 11 \rangle \langle 0, 8, 16, 15 \rangle \langle 8, 2, 9, 17 \rangle \langle 6, 2, 14, 13 \rangle \langle 6, 8, 12, 10 \rangle \langle 4, 3, 17, 15 \rangle$ $\langle 6, 3, 9, 18 \rangle \langle 4, 1, 12, 18 \rangle \langle 3, 5, 13, 12 \rangle \langle 8, 1, 19, 14 \rangle \langle 7, 0, 11, 12 \rangle \langle 1, 7, 9, 13 \rangle \langle 0, 5, 17, 18 \rangle \langle 6, 7, 17, 16 \rangle \langle 2, 7, 19, 18 \rangle$ $\langle 7, 5, 14, 15 \rangle \langle 0, 4, 9, 19 \rangle \langle 1, 2, 10, 15 \rangle \langle 4, 5, 10, 16 \rangle$									
(9, 13)	$\langle 6, 4, 13, 17 \rangle \langle 3, 2, 17, 9 \rangle \langle 0, 1, 9, 10 \rangle \langle 6, 3, 18, 15 \rangle \langle 5, 0, 16, 17 \rangle \langle 2, 5, 18, 13 \rangle \langle 7, 1, 18, 20 \rangle \langle 2, 1, 12, 15 \rangle \langle 2, 4, 16, 14 \rangle$ $\langle 1, 5, 19, 21 \rangle \langle 6, 7, 9, 12 \rangle \langle 8, 7, 13, 16 \rangle \langle 8, 2, 20, 19 \rangle \langle 0, 3, 19, 13 \rangle \langle 4, 5, 20, 9 \rangle \langle 8, 0, 18, 12 \rangle \langle 7, 3, 14, 21 \rangle \langle 7, 5, 15, 10 \rangle$ $\langle 7, 4, 19, 11 \rangle \langle 1, 3, 16, 11 \rangle \langle 6, 8, 10, 11 \rangle \langle 6, 0, 20, 14 \rangle \langle 3, 4, 12, 10 \rangle \langle 1, 8, 14, 17 \rangle \langle 0, 2, 11, 21 \rangle \langle 4, 8, 15, 21 \rangle$									
(9, 15)	$\langle 0, 1, 9, 10 \rangle \langle 0, 2, 11, 12 \rangle \langle 1, 2, 15, 13 \rangle \langle 6, 2, 22, 23 \rangle \langle 0, 5, 17, 18 \rangle \langle 0, 6, 19, 20 \rangle \langle 0, 4, 16, 15 \rangle \langle 2, 3, 9, 21 \rangle \langle 8, 1, 17, 23 \rangle$ $\langle 4, 7, 17, 13 \rangle \langle 3, 4, 23, 19 \rangle \langle 5, 8, 16, 22 \rangle \langle 2, 5, 14, 20 \rangle \langle 2, 7, 16, 19 \rangle \langle 2, 4, 10, 18 \rangle \langle 3, 6, 17, 15 \rangle \langle 4, 8, 20, 21 \rangle \langle 1, 3, 11, 22 \rangle$ $\langle 0, 3, 14, 13 \rangle \langle 3, 5, 10, 12 \rangle \langle 1, 5, 19, 21 \rangle \langle 1, 4, 14, 12 \rangle \langle 4, 5, 9, 11 \rangle \langle 7, 8, 14, 11 \rangle \langle 7, 0, 21, 22 \rangle \langle 6, 7, 9, 12 \rangle \langle 6, 8, 10, 13 \rangle$ $\langle 5, 7, 23, 15 \rangle \langle 3, 7, 18, 20 \rangle \langle 1, 6, 16, 18 \rangle$									
(9, 17)	$\langle 8, 7, 15, 17 \rangle \langle 2, 6, 18, 9 \rangle \langle 0, 5, 14, 17 \rangle \langle 4, 0, 16, 19 \rangle \langle 1, 8, 11, 25 \rangle \langle 1, 0, 18, 24 \rangle \langle 1, 5, 16, 9 \rangle \langle 2, 3, 16, 24 \rangle \langle 7, 0, 21, 10 \rangle$ $\langle 4, 8, 14, 9 \rangle \langle 5, 6, 19, 10 \rangle \langle 8, 5, 12, 18 \rangle \langle 3, 4, 15, 13 \rangle \langle 3, 7, 19, 25 \rangle \langle 0, 6, 25, 15 \rangle \langle 3, 1, 17, 20 \rangle \langle 5, 7, 24, 13 \rangle \langle 8, 0, 13, 20 \rangle$ $\langle 3, 6, 11, 14 \rangle \langle 2, 0, 11, 12 \rangle \langle 4, 6, 22, 17 \rangle \langle 8, 2, 10, 22 \rangle \langle 8, 6, 24, 23 \rangle \langle 1, 2, 15, 19 \rangle \langle 4, 1, 12, 10 \rangle \langle 7, 6, 12, 16 \rangle \langle 3, 0, 23, 9 \rangle$ $\langle 7, 4, 18, 11 \rangle \langle 3, 5, 22, 21 \rangle \langle 4, 5, 20, 25 \rangle \langle 6, 1, 21, 13 \rangle \langle 2, 4, 23, 21 \rangle \langle 7, 1, 22, 23 \rangle \langle 2, 7, 14, 20 \rangle$									

*Proof:* The upper bound  $T(2, 5; 2, 7; 6) \leq 6$  can be found in [3]. Codes achieving the upper bounds are constructed as follows.

For  $3 \leq n_1 \leq 9$ , let  $X = \{0, 1, 2, \dots, n_1 + n_2 - 1\}$ .  $X$  can be partitioned as  $X = X_1 \cup X_2$  with  $X_1 = \{0, 1, \dots, n_1 - 1\}$  and  $X_2 = \{n_1, n_1 + 1, \dots, n_1 + n_2 - 1\}$ . The desired codes are constructed on  $X$  and the codewords are listed in Table I.

For  $n_1 \in \{13, 17, 21, 25, 29, 33, 37\}$ , the codes are constructed in the Appendix.

For  $n_1 \in \{11, 15, 19\}$  and  $n_1 \leq n_2 \leq 2n_1 - 3$ , take an HSAS( $n_2, n_1; 3, 3$ ) from the Appendix and fill in the hole with an SAS\*(3, 3) (which is equivalent to an MCWC(2, 3; 2, 3; 6) and has been constructed above) to obtain an SAS\*( $n_2, n_1$ ). According to Proposition 6.3, that is equivalent to an MCWC(2,  $n_1; 2, n_2; 6$ ) of size  $\lfloor \frac{n_2(n_1-1)}{4} \rfloor$ , as desired. For  $n_1 \in \{11, 15, 19\}$  and  $n_2 = 2n_1 - 1$ , we proceed similarly; take an HSAS( $n_2, n_1; 5, 3$ ) from the Appendix and fill in the hole with an SAS\*(5, 3) (which is equivalent to an MCWC(2, 5; 2, 3; 6) and has been constructed above). ■

**Lemma 6.8:** Let  $t$  be a positive integer with  $2t + 1 \geq 21$  and  $2t + 1 \notin \{23, 27, 29, 33, 39, 43, 51, 59, 75, 83, 87, 95, 99, 107, 139, 179\}$ . Let  $n_1 = 4t + 1$  or  $4t + 3$ ,  $n_1 \leq n_2 \leq 2n_1 - 1$  and  $n_2$  be odd. Then  $T(2, n_1; 2, n_2; 6) = \lfloor \frac{n_2(n_1-1)}{4} \rfloor$ .

*Proof:* According to Propositions 6.2 and 6.3, we only need to construct the corresponding SAS( $n_2, n_1$ ) when  $n_1 \equiv 1 \pmod{4}$  or SAS\*( $n_2, n_1$ ) when  $n_1 \equiv 3 \pmod{4}$ .

For each given  $t$  and  $2t + 1 \notin \{71, 111, 113, 115, 119\}$ , take a  $(2t + 1, \{5, 7, 9\}, 1)$ -PBD from Theorem 2.6, and remove one point to obtain a  $\{5, 7, 9\}$ -GDD of type  $4^i 6^j 8^k$  with  $4i + 6j + 8k = 2t$ . Assign each point with weights (4, 2) or (2, 2) and apply Construction 6.4; the input SFSs of type  $(4, 2)^a (2, 2)^b$  with  $a + b \in \{5, 7, 9\}$  are constructed in the Appendix. Then we can get an SFS of type

$$(8, 8)^{i_8} (10, 8)^{i_{10}} \dots (16, 8)^{i_{16}} (12, 12)^{j_{12}} \dots (24, 12)^{j_{24}} (16, 16)^{k_{16}} \dots (32, 16)^{k_{32}},$$

for any nonnegative integers  $i_8, i_{10}, \dots, i_{16}, j_{12}, \dots, j_{24}, k_{16}, \dots, k_{32}$  with

$$i_8 + i_{10} + \dots + i_{16} = i$$

$$j_{12} + j_{14} + \dots + j_{24} = j$$

$$k_{16} + k_{18} + \dots + k_{32} = k.$$

Now, we can fill the holes of the SFS in three ways:

- 1) Add a new row and a new column and apply Construction 6.6 (2) with ‘ $e = 1$ ’ and ‘ $w = 1$ ’; the input HSAS( $r, v; 1, 1$ ) (i.e. SAS( $r, v$ )) with  $v \in \{9, 13, 17\}$  and  $v \leq r \leq 2v - 1$  come from Lemma 6.7. Then we get an SAS( $s, 4t + 1; 1, 1$ ) with  $4t + 1 \leq s \leq 8t + 1$ , as desired;



- 2) Add three new rows and three new columns and apply Construction 6.6 (1) with ‘ $e = 3$ ’ and ‘ $w = 3$ ’; the input HSAS( $r, v; 3, 3$ ) with  $v \in \{11, 15, 19\}$  and  $v \leq r \leq 2v - 3$  are constructed in the Appendix. We get an HSAS( $s, 4t + 3; 3, 3$ ) with  $4t + 3 \leq s \leq 8t + 3$ . Then fill in the hole with an SAS( $3, 3$ ) constructed in Lemma 6.7 to obtain the desired SAS\*( $s, 4t + 3$ ) with  $4t + 3 \leq s \leq 8t + 3$ .
- 3) When the SFS has type  $(16, 8)^i(24, 12)^j(32, 16)^k$ , add five new rows and five new columns and apply Construction 6.6 (1) with ‘ $e = 5$ ’ and ‘ $w = 3$ ’; the input HSAS( $r, v; 5, 3$ ) with  $(r, v) \in \{(21, 11), (29, 15), (37, 19)\}$  are constructed in the Appendix. We get an HSAS( $8t + 5, 4t + 3; 5, 3$ ). Then fill in the hole with an SAS\*( $5, 3$ ) constructed in Lemma 6.7 to obtain the desired SAS\*( $8t + 5, 4t + 3$ ).

For  $2t + 1 = 71$ , take a TD(9, 8) from Theorem 2.7 and truncate one of its group to six points to obtain an  $\{8, 9\}$ -GDD of type  $8^8 6^1$ , noting that  $8 \times 8 + 6 = 70 = 2t$ . Then proceed similarly as above, we can obtain the desired SAS( $s, 4t + 1$ ) and SAS\*( $s, 4t + 3$ ). Here the additional input SFSs of type  $(4, 2)^a(2, 2)^{8-a}$  with  $0 \leq a \leq 8$  are constructed in the Appendix.

For  $2t + 1 \in \{111, 113, 115, 119\}$ , take a  $\{7, 9\}$ -GDD of type  $8^{15}$  from [10, Part 4, Corollary 2.44] and truncate the last two groups to obtain  $\{5, 6, 7, 8, 9\}$ -GDDs of types  $8^{13} 6^1$ ,  $8^{14}$ ,  $8^{13} 6^1 4^1$  and  $8^{14} 6^1$ , respectively. Then proceed similarly as above, we can obtain the desired SASs and SAS\*s. Here the additional input SFSs of type  $(4, 2)^a(2, 2)^{6-a}$  with  $0 \leq a \leq 6$  are constructed in the Appendix. ■

*Remark 6.9:* In the proof of Lemma 6.8, we have constructed HSAS( $s, 4t + 3; 3, 3$ ) with  $4t + 3 \leq s \leq 8t + 3$  and HSAS( $8t + 5, 4t + 3; 5, 3$ ). These HSASs will be used in later constructions.

*Lemma 6.10:* Let  $t$  be a positive integer with  $2t + 1 \in \{39, 43, 51, 59, 75, 99\}$ . Let  $n_1 = 4t + 1$  or  $4t + 3$ ,  $n_1 \leq n_2 \leq 2n_1 - 1$  and  $n_2$  be odd. Then  $T(2, n_1; 2, n_2; 6) = \lfloor \frac{n_2(n_1-1)}{4} \rfloor$ .

*Proof:* For  $2t + 1 = 39$ , take a  $\{5, 7\}$ -GDD of type  $6^6 2^1$  from [10, Part 4, Example 2.51], noting that  $6 \times 6 + 2 = 2t$ . Assign each point with weights  $(4, 2)$  or  $(2, 2)$  and apply Construction 6.4. Then we can get an SFS of type

$$(12, 12)^{i_{12}}(14, 12)^{i_{14}} \dots (24, 12)^{i_{24}}(4, 4)^{j_4}(8, 4)^{j_8},$$

for any nonnegative integers  $i_{12}, i_{14}, \dots, i_{24}, j_4, j_8$  with  $i_{12} + i_{14} + \dots + i_{24} = 6$  and  $j_4 + j_8 = 1$ . Now, we can fill the holes of the SFS in three ways:

- 1) Add a new row and a new column and apply Construction 6.6 (2) with ‘ $e = 1$ ’ and ‘ $w = 1$ ’; the input HSAS( $r, 13; 1, 1$ ) (i.e. SAS( $r, 13$ )) with  $13 \leq r \leq 25$ , SAS( $5, 5$ ) and SAS( $9, 5$ ) come from Lemma 6.7. Then we get an SAS( $s, 77$ ) with  $77 \leq s \leq 153$ , as desired.
- 2) Add three new rows and three new columns and apply Construction 6.6 (3) with ‘ $e = 3$ ’ and ‘ $w = 3$ ’; the input HSAS( $r, 15; 3, 3$ ) with  $15 \leq r \leq 27$  are constructed in the Appendix and the input SAS\*( $7, 7$ ) and SAS\*( $11, 7$ ) come from Lemma 6.7. Then we get an SAS\*( $s, 79$ ) with  $79 \leq s \leq 155$ .
- 3) When the SFS has type  $(24, 12)^6(8, 4)^1$ , add five new rows and five new columns and apply Construction 6.6 (3) with ‘ $e = 5$ ’ and ‘ $w = 3$ ’; the input HSAS( $29, 15; 5, 3$ ) is constructed in the Appendix and the input SAS\*( $13, 7$ ) comes from Lemma 6.7. Then we get an SAS\*( $157, 79$ ), as desired.

For  $2t + 1 \in \{43, 51, 59, 75, 99\}$ , we start with  $\{5, 6, 7, 8, 9\}$ -GDDs of types  $8^5 2^1$ ,  $8^6 2^1$ ,  $8^7 2^1$ ,  $8^9 2^1$ , and  $8^{12} 2^1$ , respectively, which will be constructed below. Proceed as above to obtain the desired SASs and SAS\*s; here we fill in the holes of the SFS with SAS( $r, 17$ ) (see Lemma 6.7), HSAS( $r, 19; 3, 3$ ) (see the Appendix) and HSAS( $37, 19; 5, 3$ ) (see the Appendix). The  $\{5, 6, 7, 8, 9\}$ -GDDs are constructed as follows. For the types  $8^5 2^1$ ,  $8^6 2^1$  and  $8^7 2^1$ , take a TD(9, 8) from Theorem 2.7 and truncate the last four groups. For the type  $8^9 2^1$ , take a TD(9, 9) from Theorem 2.7 and remove one point to redefine the groups to obtain a  $\{9\}$ -GDD of type  $8^{10}$ . Then truncate the last group. For the type  $8^{12} 2^1$ , take a  $\{9\}$ -GDD of type  $8^{15} 16^1$  from [10, Part 4, Corollary 2.44] and truncate the last four groups. ■

*Lemma 6.11:* Let  $t$  be a positive integer. If  $2t + 1 \in \{107, 139, 179\}$ . Let  $n_1 = 4t + 1$  or  $4t + 3$ ,  $n_1 \leq n_2 \leq 2n_1 - 1$  and  $n_2$  be odd. Then  $T(2, n_1; 2, n_2; 6) = \lfloor \frac{n_2(n_1-1)}{4} \rfloor$ .

*Proof:* For  $2t + 1 = 107$ , take a TD(6, 20) from Theorem 2.7 and truncate the last group to six points to obtain a  $\{5, 6\}$ -GDD of type  $20^5 6^1$ . Assign each point with weights  $(4, 2)$  or  $(2, 2)$  and apply Construction 6.4. Then we can get an SFS of type

$$(40, 40)^{i_{40}}(42, 40)^{i_{42}} \dots (80, 40)^{i_{80}}(12, 12)^{j_{12}}(14, 12)^{j_{14}} \dots (24, 12)^{j_{24}},$$

for any nonnegative integers  $i_{40}, i_{42}, \dots, i_{80}, j_{12}, \dots, j_{24}$  with  $i_{40} + i_{42} + \dots + i_{80} = 5$  and  $j_{12} + j_{14} + \dots + j_{24} = 1$ . Now, we can fill the holes of the SFS in three ways:

- 1) Add a new row and a new column and apply Construction 6.6 (2) with ‘ $e = 1$ ’ and ‘ $w = 1$ ’; the input HSASs and SASs come from Lemmas 6.7–6.8. Then we get an SAS( $s, 213$ ) with  $213 \leq s \leq 425$ , as desired.
- 2) Add three new rows and three new columns and apply Construction 6.6 (3) with ‘ $e = 3$ ’ and ‘ $w = 3$ ’; the input HSAS( $r, 43; 3, 3$ ) with  $43 \leq r \leq 83$  are constructed in the proof of Lemma 6.8 and the input SAS\*( $r, 15$ ) comes from Lemma 6.7. Then we get an SAS\*( $s, 215$ ) with  $215 \leq s \leq 427$ .
- 3) When the SFS has type  $(80, 40)^5(24, 12)^1$ , add five new rows and five new columns and apply Construction 6.6 (3) with ‘ $e = 5$ ’ and ‘ $w = 3$ ’; the input HSAS( $85, 43; 5, 3$ ) and SAS\*( $29, 15$ ) come from Lemmas 6.7–6.8. Then we get an SAS\*( $429, 215$ ), as desired.



For  $2t + 1 = 139$  or  $179$ , take a  $\text{TD}(8, 24)$  from Theorem 2.7 and truncate the last three groups to obtain  $\{5, 6, 7, 8, 9\}$ -GDDs of types  $24^5 6^3$  or  $24^7 6^1 4^1$ . Then proceed similarly as above to obtain the desired SASs and SAS\*s; the input HSASs, SASs and SAS\*s all come from Lemma 6.7. ■

**Lemma 6.12:** Let  $t$  be a positive integer with  $2t + 1 \in \{83, 87, 95\}$ . Let  $n_1 = 4t + 1$ ,  $n_1 \leq n_2 \leq 2n_1 - 1$  and  $n_2$  be odd. Then  $T(2, n_1; 2, n_2; 6) = \frac{n_2(n_1-1)}{4}$ .

**Proof:** Take a  $\text{TD}(6, 16)$  from Theorem 2.7 and truncate the last group to obtain  $\{5, 6\}$ -GDDs of types  $16^5 2^1$ ,  $16^5 6^1$  or  $16^5 14^1$ , respectively. Then proceed similarly as above to obtain the desired SASs; the input  $\text{SAS}(s, v)$  with  $s \in \{5, 13, 29, 33\}$  all come from Lemma 6.7. ■

Combining the above lemmas, we get the following result.

**Theorem 6.13:** Let  $n_1, n_2$  be two odd integers with  $0 < n_1 \leq n_2 \leq 2n_1 - 1$ . Then  $T(2, n_1; 2, n_2; 6) = \lfloor \frac{n_2(n_1-1)}{4} \rfloor$ , except for  $(n_1, n_2) = (5, 7)$ , and except possibly for  $n_1 \in \{23, 27, 31, 35, 39, 45, 47, 53, 55, 57, 59, 65, 67, 165, 175, 191\}$ .

## VII. CONCLUSIONS

In this paper, we consider the bounds and constructions of MCWCs. For the upper bound, we use three different approaches to improve the generalised Johnson bounds mentioned in [5]. For the lower bound, we derive two asymptotic lower bounds, the first is from the technique of concatenation and the second is from the Gilbert-Varshamov type bound. A comparison between these two bounds is also given. For the constructions, by establishing the connections between some combinatorial structures and MCWCs, several new combinatorial constructions for MCWCs are given. We obtain the asymptotic existence result of two classes of optimal MCWCs and construct a class of optimal MWCWs which are open in [7]. As consequences, the Johnson-type bounds are shown to be asymptotically exact for MCWCs with distances  $2 \sum_{i=1}^m w_i - 2$  or  $2mw - w$ . The maximum sizes of MCWCs with total weight less than or equal to four are determined almost completely.

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## VIII. APPENDIX

### A. Small MCWC( $2, n_1; 2, n_2; 6$ ) for $n_1 \equiv 1 \pmod{4}$ and $13 \leq n_1 \leq 37$

**Lemma 8.1:**  $T(2, 13; 2, n; 6) = 3n$  for each odd  $n$  and  $13 \leq n \leq 25$ .

**Proof:** Let  $X_1 = (\mathbb{Z}_3 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $13 \leq n \leq 17$ , let  $X_2 = (\mathbb{Z}_3 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-12\})$ ; for  $19 \leq n \leq 23$ , let  $X_2 = (\mathbb{Z}_3 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-18\})$ ; for  $n = 25$ , let  $X_2 = (\mathbb{Z}_3 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ .

Denote  $X = X_1 \cup X_2$ . The desired codes of size  $3n$  are constructed on  $\mathbb{Z}_2^X$ . The codewords are obtained by developing the following base codewords under the action of the cyclic group  $\mathbb{Z}_3$ , where the points  $\infty$  and  $a_i$  are fixed.

$n = 13$ :

$$\begin{array}{cccccccc} \langle \infty, 0_0, 0_5, 0_4 \rangle & \langle \infty, 0_3, 2_6, 2_7 \rangle & \langle 1_1, 1_0, 2_4, a_1 \rangle & \langle 2_2, 2_3, 2_5, a_1 \rangle & \langle 1_0, 2_2, 0_7, 2_6 \rangle & \langle 1_1, 2_2, 1_5, 1_6 \rangle & \langle 0_0, 2_1, 0_7, 2_4 \rangle & \langle 0_0, 2_3, 0_6, 1_5 \rangle \\ \langle 2_1, 2_2, 1_4, 0_5 \rangle & \langle 1_2, 2_3, 2_4, 2_6 \rangle & \langle 2_0, 0_1, 0_7, 1_6 \rangle & \langle 1_2, 0_3, 0_7, 1_4 \rangle & \langle 0_1, 1_3, 2_5, 2_7 \rangle & & & \end{array}$$

$n = 15$ :

$$\begin{array}{cccccccc} \langle \infty, 2_1, 2_7, 2_6 \rangle & \langle \infty, 0_3, 0_5, 2_4 \rangle & \langle 2_3, 2_2, 0_5, a_1 \rangle & \langle 2_1, 2_0, 1_7, a_1 \rangle & \langle 2_2, 0_3, 0_7, a_2 \rangle & \langle 1_0, 2_1, 0_4, a_2 \rangle & \langle 0_0, 2_1, 0_5, a_3 \rangle & \langle 0_3, 1_2, 1_4, a_3 \rangle \\ \langle 0_1, 2_3, 2_6, 2_4 \rangle & \langle 2_1, 0_2, 2_4, 0_7 \rangle & \langle 1_0, 1_3, 0_6, 2_7 \rangle & \langle 0_1, 0_3, 1_6, 2_5 \rangle & \langle 1_2, 2_0, 2_4, 0_6 \rangle & \langle 0_2, 0_0, 2_5, 0_6 \rangle & \langle 0_0, 1_2, 1_5, 0_7 \rangle & \end{array}$$

$n = 17$ :

$$\begin{array}{cccccccc} \langle \infty, 1_1, 1_7, 1_6 \rangle & \langle \infty, 0_3, 1_4, 2_5 \rangle & \langle 1_2, 1_3, 0_6, a_1 \rangle & \langle 0_0, 0_1, 1_4, a_1 \rangle & \langle 1_0, 0_1, 1_6, a_2 \rangle & \langle 1_3, 0_2, 2_5, a_2 \rangle & \langle 0_1, 2_0, 2_5, a_3 \rangle & \langle 2_3, 0_2, 1_7, a_3 \rangle \\ \langle 1_1, 0_3, 0_4, a_4 \rangle & \langle 0_0, 1_2, 2_6, a_4 \rangle & \langle 2_1, 0_3, 2_4, a_5 \rangle & \langle 0_2, 1_0, 2_7, a_5 \rangle & \langle 1_3, 0_0, 1_5, 2_7 \rangle & \langle 1_2, 0_1, 1_5, 1_7 \rangle & \langle 0_0, 0_3, 0_7, 1_6 \rangle & \langle 1_1, 0_2, 1_5, 0_6 \rangle \\ \langle 0_0, 0_2, 0_4, 2_4 \rangle & & & & & & & \end{array}$$

$n = 19$ :

$$\begin{array}{cccccccc} \langle \infty, 0_0, 0_4, 0_5 \rangle & \langle \infty, 0_1, 0_6, 0_7 \rangle & \langle \infty, 0_2, 0_8, 0_9 \rangle & \langle 0_0, 0_1, 1_4, a_1 \rangle & \langle 0_2, 0_3, 0_5, a_1 \rangle & \langle 0_0, 1_0, 2_5, 0_6 \rangle & \langle 0_3, 1_3, 0_4, 0_8 \rangle & \langle 0_0, 1_3, 2_7, 2_9 \rangle \\ \langle 0_0, 0_2, 1_7, 1_8 \rangle & \langle 0_1, 1_1, 0_5, 0_8 \rangle & \langle 0_1, 0_2, 2_4, 1_5 \rangle & \langle 0_1, 1_2, 1_6, 0_9 \rangle & \langle 0_2, 1_2, 2_6, 1_4 \rangle & \langle 0_2, 1_3, 2_5, 0_7 \rangle & \langle 0_2, 2_3, 2_7, 1_9 \rangle & \langle 0_0, 1_1, 0_7, 2_8 \rangle \\ \langle 0_0, 2_1, 2_4, 0_9 \rangle & \langle 0_0, 2_3, 1_6, 0_8 \rangle & \langle 0_1, 2_3, 2_6, 2_9 \rangle & & & & & \end{array}$$

$n = 21$ :

$$\begin{array}{cccccccc} \langle \infty, 1_1, 1_7, 1_6 \rangle & \langle \infty, 0_2, 1_8, 0_9 \rangle & \langle \infty, 1_0, 1_4, 1_5 \rangle & \langle 2_1, 2_0, 0_7, a_1 \rangle & \langle 0_2, 0_3, 0_5, a_1 \rangle & \langle 1_3, 0_1, 1_4, a_2 \rangle & \langle 0_0, 0_2, 1_5, a_2 \rangle & \langle 2_0, 1_1, 1_4, a_3 \rangle \\ \langle 2_3, 1_2, 0_7, a_3 \rangle & \langle 0_2, 1_2, 2_6, 0_4 \rangle & \langle 0_0, 1_2, 2_9, 2_7 \rangle & \langle 1_0, 1_3, 1_7, 2_9 \rangle & \langle 0_1, 0_3, 2_7, 1_8 \rangle & \langle 0_0, 1_0, 2_6, 0_8 \rangle & \langle 0_3, 2_3, 1_4, 2_8 \rangle & \langle 1_1, 2_2, 1_5, 0_4 \rangle \\ \langle 2_1, 1_3, 1_6, 0_6 \rangle & \langle 0_0, 1_3, 0_9, 2_5 \rangle & \langle 1_1, 1_2, 1_8, 0_8 \rangle & \langle 1_1, 2_1, 0_9, 0_5 \rangle & \langle 0_2, 2_3, 0_6, 2_9 \rangle & & & \end{array}$$

$n = 23$ :

$$\begin{array}{cccccccc} \langle \infty, 0_0, 0_5, 0_4 \rangle & \langle \infty, 1_2, 0_8, 0_9 \rangle & \langle \infty, 1_1, 2_6, 0_7 \rangle & \langle 0_3, 0_2, 0_5, a_1 \rangle & \langle 1_0, 1_1, 2_4, a_1 \rangle & \langle 2_1, 1_0, 2_5, a_2 \rangle & \langle 0_3, 2_2, 2_4, a_2 \rangle & \langle 2_0, 1_1, 2_7, a_3 \rangle \\ \langle 2_3, 0_2, 0_6, a_3 \rangle & \langle 0_0, 0_2, 2_5, a_4 \rangle & \langle 0_3, 0_1, 0_6, a_4 \rangle & \langle 1_1, 2_3, 0_4, a_5 \rangle & \langle 0_2, 2_0, 2_6, a_5 \rangle & \langle 0_0, 2_2, 0_8, 1_7 \rangle & \langle 0_0, 2_0, 1_6, 1_8 \rangle & \langle 0_0, 2_3, 0_9, 2_4 \rangle \\ \langle 0_3, 1_3, 2_8, 2_5 \rangle & \langle 0_1, 2_3, 2_9, 0_7 \rangle & \langle 0_1, 1_1, 2_5, 1_8 \rangle & \langle 0_1, 1_2, 2_6, 1_9 \rangle & \langle 0_0, 0_3, 2_7, 2_9 \rangle & \langle 1_2, 2_1, 1_8, 2_9 \rangle & \langle 0_2, 1_2, 2_4, 1_7 \rangle & \end{array}$$

$n = 25$ :

$$\begin{array}{cccccccc} \langle \infty, 0_3, 0_{11}, 0_7 \rangle & \langle \infty, 2_0, 2_5, 1_9 \rangle & \langle \infty, 0_2, 0_6, 1_8 \rangle & \langle \infty, 0_1, 1_{10}, 1_4 \rangle & \langle 0_0, 0_2, 0_7, a_1 \rangle & \langle 0_1, 1_3, 0_6, a_1 \rangle & \langle 2_2, 0_1, 0_5, 1_7 \rangle & \langle 1_3, 1_0, 0_8, 2_5 \rangle \\ \langle 2_0, 0_3, 0_8, 1_7 \rangle & \langle 0_3, 1_2, 1_{10}, 0_{10} \rangle & \langle 0_0, 2_1, 1_6, 0_9 \rangle & \langle 0_0, 1_2, 2_6, 0_6 \rangle & \langle 2_0, 1_0, 0_4, 2_{11} \rangle & \langle 0_0, 0_1, 0_4, 2_{10} \rangle & \langle 0_2, 1_2, 1_9, 2_{11} \rangle & \langle 2_1, 0_2, 0_8, 0_{11} \rangle \\ \langle 1_2, 2_0, 0_9, 2_{10} \rangle & \langle 1_0, 0_3, 0_5, 2_7 \rangle & \langle 0_1, 2_1, 2_7, 2_9 \rangle & \langle 2_3, 0_3, 1_4, 1_9 \rangle & \langle 2_2, 2_1, 1_4, 1_5 \rangle & \langle 0_0, 1_1, 1_{10}, 0_8 \rangle & \langle 1_3, 1_1, 2_6, 0_{11} \rangle & \langle 2_1, 1_3, 0_5, 2_{11} \rangle \\ \langle 0_2, 1_3, 1_4, 2_8 \rangle & & & & & & & \end{array}$$

■

**Lemma 8.2:**  $T(2, 17; 2, n; 6) = 4n$  for each odd  $n$  and  $17 \leq n \leq 33$ .

*Proof:* Let  $X_1 = (\mathbb{Z}_4 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $17 \leq n \leq 23$ , let  $X_2 = (\mathbb{Z}_4 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-16\})$ ; for  $25 \leq n \leq 31$ , let  $X_2 = (\mathbb{Z}_4 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-24\})$ ; for  $n = 33$ , let  $X_2 = (\mathbb{Z}_4 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ . Denote  $X = X_1 \cup X_2$ . The codes of size  $4n$  are constructed on  $\mathbb{Z}_2^X$  and the base codewords are listed as follows.

$n = 17$ :

$$\begin{array}{cccccccc} \langle \infty, 1_0, 1_4, 1_5 \rangle & \langle \infty, 2_1, 1_7, 2_6 \rangle & \langle 3_0, 3_1, 0_4, a_1 \rangle & \langle 3_2, 2_3, 3_5, a_1 \rangle & \langle 2_0, 2_3, 1_7, 1_5 \rangle & \langle 0_2, 2_3, 0_6, 2_5 \rangle & \langle 0_1, 1_1, 1_5, 0_4 \rangle & \langle 0_2, 1_2, 2_4, 1_7 \rangle \\ \langle 1_1, 1_3, 3_5, 0_6 \rangle & \langle 1_1, 0_2, 3_6, 3_4 \rangle & \langle 1_0, 3_1, 1_7, 0_6 \rangle & \langle 0_3, 1_3, 1_6, 2_4 \rangle & \langle 0_0, 1_0, 2_5, 3_4 \rangle & \langle 2_0, 3_3, 0_7, 3_7 \rangle & \langle 0_1, 2_2, 1_7, 3_5 \rangle & \langle 0_0, 3_2, 0_6, 1_6 \rangle \\ \langle 0_2, 0_3, 0_4, 2_7 \rangle & & & & & & & \end{array}$$

$n = 19$ :

$$\begin{array}{cccccccc} \langle \infty, 3_0, 1_4, 2_5 \rangle & \langle \infty, 3_1, 2_6, 0_7 \rangle & \langle 1_0, 2_1, 1_7, a_1 \rangle & \langle 0_3, 1_2, 0_6, a_1 \rangle & \langle 2_0, 3_3, 3_5, a_2 \rangle & \langle 0_1, 1_2, 0_4, a_2 \rangle & \langle 0_2, 0_3, 1_4, a_3 \rangle & \langle 1_1, 2_0, 0_5, a_3 \rangle \\ \langle 0_0, 2_3, 0_5, 3_7 \rangle & \langle 3_3, 2_2, 3_7, 2_5 \rangle & \langle 0_0, 0_3, 1_6, 0_4 \rangle & \langle 2_0, 3_2, 3_4, 1_6 \rangle & \langle 3_2, 3_1, 1_4, 3_7 \rangle & \langle 1_1, 0_2, 3_7, 3_5 \rangle & \langle 0_1, 3_1, 0_6, 0_5 \rangle & \langle 2_3, 3_3, 1_4, 1_6 \rangle \\ \langle 2_1, 2_0, 0_6, 1_4 \rangle & \langle 1_2, 0_2, 2_5, 1_6 \rangle & \langle 0_0, 3_3, 1_7, 2_7 \rangle & & & & & \end{array}$$

$n = 21$ :

$$\begin{array}{cccccccc} \langle \infty, 2_1, 2_6, 2_7 \rangle & \langle \infty, 2_0, 0_5, 2_4 \rangle & \langle 3_2, 3_3, 3_5, a_1 \rangle & \langle 0_0, 0_1, 1_6, a_1 \rangle & \langle 0_3, 3_2, 1_5, a_2 \rangle & \langle 3_1, 1_0, 3_4, a_2 \rangle & \langle 3_2, 1_3, 3_6, a_3 \rangle & \langle 0_0, 1_1, 3_4, a_3 \rangle \\ \langle 2_3, 3_2, 0_4, a_4 \rangle & \langle 1_0, 0_1, 2_5, a_4 \rangle & \langle 2_3, 0_1, 3_7, a_5 \rangle & \langle 1_0, 1_2, 0_5, a_5 \rangle & \langle 2_1, 1_3, 3_7, 3_5 \rangle & \langle 0_2, 0_1, 3_6, 3_4 \rangle & \langle 0_1, 1_3, 2_6, 0_5 \rangle & \langle 1_2, 0_1, 2_7, 1_4 \rangle \\ \langle 1_0, 3_2, 1_6, 3_7 \rangle & \langle 3_0, 2_2, 3_5, 0_4 \rangle & \langle 2_0, 3_2, 1_7, 0_6 \rangle & \langle 1_3, 2_3, 1_6, 2_4 \rangle & \langle 0_0, 1_3, 0_7, 1_7 \rangle & & & \end{array}$$

$n = 23$ :

$$\begin{array}{cccccccc} \langle \infty, 3_0, 3_5, 3_4 \rangle & \langle \infty, 3_1, 1_7, 0_6 \rangle & \langle 1_2, 1_3, 2_6, a_1 \rangle & \langle 2_0, 2_1, 3_4, a_1 \rangle & \langle 2_1, 1_0, 0_4, a_2 \rangle & \langle 3_3, 2_2, 2_5, a_2 \rangle & \langle 0_0, 1_3, 2_5, a_3 \rangle & \langle 0_1, 2_2, 0_7, a_3 \rangle \\ \langle 1_0, 0_1, 2_5, a_4 \rangle & \langle 2_3, 3_2, 3_7, a_4 \rangle & \langle 2_3, 2_1, 1_4, a_5 \rangle & \langle 2_2, 1_0, 1_6, a_5 \rangle & \langle 0_1, 1_3, 3_5, a_6 \rangle & \langle 0_0, 2_2, 2_6, a_6 \rangle & \langle 3_0, 3_2, 2_5, a_7 \rangle & \langle 2_3, 0_1, 1_7, a_7 \rangle \\ \langle 0_2, 0_1, 3_7, 0_4 \rangle & \langle 0_3, 3_3, 1_4, 2_6 \rangle & \langle 3_0, 2_2, 0_6, 3_7 \rangle & \langle 1_1, 0_1, 3_6, 1_5 \rangle & \langle 1_3, 2_0, 3_7, 1_6 \rangle & \langle 2_3, 0_0, 2_7, 2_4 \rangle & \langle 0_2, 1_2, 3_4, 2_5 \rangle & \end{array}$$

$n = 25$ :

$$\begin{array}{cccccccc} \langle \infty, 3_1, 3_7, 1_6 \rangle & \langle \infty, 2_0, 2_4, 2_5 \rangle & \langle \infty, 0_2, 0_9, 0_8 \rangle & \langle 3_3, 0_2, 3_6, a_1 \rangle & \langle 0_0, 0_1, 1_4, a_1 \rangle & \langle 0_3, 1_3, 2_4, 2_9 \rangle & \langle 2_2, 0_1, 0_6, 3_9 \rangle & \langle 0_1, 1_1, 1_5, 0_4 \rangle \\ \langle 1_2, 0_0, 2_8, 2_6 \rangle & \langle 2_0, 1_0, 3_5, 0_4 \rangle & \langle 0_3, 1_0, 1_7, 0_8 \rangle & \langle 1_2, 1_3, 0_4, 3_8 \rangle & \langle 2_1, 2_2, 0_4, 1_7 \rangle & \langle 1_1, 0_3, 1_8, 0_5 \rangle & \langle 3_1, 0_3, 1_5, 2_6 \rangle & \langle 2_0, 0_2, 3_9, 0_7 \rangle \\ \langle 2_0, 0_1, 1_6, 3_8 \rangle & \langle 0_0, 3_1, 0_8, 3_9 \rangle & \langle 2_1, 0_3, 0_9, 0_7 \rangle & \langle 0_0, 1_3, 3_7, 0_9 \rangle & \langle 0_0, 0_3, 3_5, 1_6 \rangle & \langle 0_1, 3_2, 1_7, 2_8 \rangle & \langle 2_2, 1_2, 2_4, 0_5 \rangle & \langle 2_0, 2_2, 2_6, 0_9 \rangle \\ \langle 0_2, 2_3, 0_5, 1_7 \rangle & & & & & & & \end{array}$$

$n = 27$ :

$$\begin{array}{cccccccc} \langle \infty, 2_0, 2_4, 2_5 \rangle & \langle \infty, 2_2, 0_8, 2_9 \rangle & \langle \infty, 0_1, 1_6, 1_7 \rangle & \langle 2_3, 2_2, 0_7, a_1 \rangle & \langle 2_1, 2_0, 3_4, a_1 \rangle & \langle 1_2, 2_3, 3_5, a_2 \rangle & \langle 2_1, 1_0, 0_6, a_2 \rangle & \langle 3_0, 1_1, 1_4, a_3 \rangle \\ \langle 2_3, 0_2, 1_5, a_3 \rangle & \langle 1_1, 3_2, 1_6, 2_9 \rangle & \langle 1_1, 2_1, 2_8, 3_5 \rangle & \langle 0_0, 0_2, 1_6, 0_7 \rangle & \langle 3_0, 2_3, 2_5, 1_6 \rangle & \langle 0_0, 2_2, 1_8, 1_7 \rangle & \langle 1_0, 3_3, 1_6, 0_9 \rangle & \langle 0_2, 3_3, 2_9, 2_4 \rangle \\ \langle 0_2, 1_2, 0_6, 1_4 \rangle & \langle 1_1, 2_0, 0_7, 3_9 \rangle & \langle 0_1, 3_3, 3_6, 3_4 \rangle & \langle 1_0, 0_0, 2_5, 0_8 \rangle & \langle 0_3, 1_3, 0_8, 2_4 \rangle & \langle 1_0, 1_3, 1_9, 0_7 \rangle & \langle 2_1, 2_3, 0_8, 2_7 \rangle & \langle 2_0, 3_2, 0_9, 0_8 \rangle \\ \langle 1_1, 3_3, 1_9, 0_8 \rangle & \langle 0_1, 1_2, 2_7, 0_5 \rangle & \langle 0_1, 3_2, 2_4, 3_5 \rangle & & & & & \end{array}$$

$n = 29$ :

$\langle \infty, 0_0, 0_4, 0_5 \rangle$	$\langle \infty, 0_1, 0_7, 0_6 \rangle$	$\langle \infty, 2_2, 2_9, 2_8 \rangle$	$\langle 2_3, 2_2, 2_5, a_1 \rangle$	$\langle 0_1, 0_0, 1_4, a_1 \rangle$	$\langle 2_1, 1_0, 0_4, a_2 \rangle$	$\langle 2_3, 1_2, 3_5, a_2 \rangle$	$\langle 2_3, 0_2, 2_6, a_3 \rangle$
$\langle 3_1, 1_0, 3_4, a_3 \rangle$	$\langle 3_3, 0_2, 3_4, a_4 \rangle$	$\langle 1_1, 2_0, 3_5, a_4 \rangle$	$\langle 0_2, 0_0, 3_5, a_5 \rangle$	$\langle 1_1, 1_3, 0_4, a_5 \rangle$	$\langle 0_0, 3_0, 2_8, 3_9 \rangle$	$\langle 0_0, 2_3, 1_9, 3_7 \rangle$	$\langle 0_3, 3_0, 3_7, 1_6 \rangle$
$\langle 0_2, 0_1, 1_8, 3_6 \rangle$	$\langle 3_1, 0_3, 2_7, 2_9 \rangle$	$\langle 3_1, 2_3, 1_8, 0_5 \rangle$	$\langle 3_1, 0_2, 1_6, 0_6 \rangle$	$\langle 3_1, 1_3, 2_8, 1_7 \rangle$	$\langle 1_3, 0_3, 1_9, 2_4 \rangle$	$\langle 0_1, 3_1, 1_9, 3_5 \rangle$	$\langle 0_2, 1_0, 2_7, 3_9 \rangle$
$\langle 1_2, 3_1, 0_7, 3_8 \rangle$	$\langle 2_0, 1_3, 3_8, 3_6 \rangle$	$\langle 0_2, 3_2, 1_9, 1_4 \rangle$	$\langle 3_0, 3_3, 2_6, 3_8 \rangle$	$\langle 0_0, 1_2, 2_5, 2_7 \rangle$			

$n = 31$ :

$\langle \infty, 2_3, 1_9, 1_5 \rangle$	$\langle \infty, 2_0, 0_6, 1_7 \rangle$	$\langle \infty, 3_2, 0_8, 3_4 \rangle$	$\langle 3_3, 3_1, 0_4, a_1 \rangle$	$\langle 3_0, 0_2, 3_5, a_1 \rangle$	$\langle 3_1, 2_0, 2_4, a_2 \rangle$	$\langle 1_3, 3_2, 1_6, a_2 \rangle$	$\langle 3_1, 1_0, 3_4, a_3 \rangle$
$\langle 1_3, 0_2, 0_6, a_3 \rangle$	$\langle 1_0, 0_1, 2_6, a_4 \rangle$	$\langle 0_2, 0_3, 0_7, a_4 \rangle$	$\langle 3_0, 0_3, 0_4, a_5 \rangle$	$\langle 2_1, 0_2, 0_5, a_5 \rangle$	$\langle 1_2, 1_1, 2_5, a_6 \rangle$	$\langle 3_0, 2_3, 3_6, a_6 \rangle$	$\langle 1_1, 3_3, 1_5, a_7 \rangle$
$\langle 1_2, 3_0, 3_7, a_7 \rangle$	$\langle 3_0, 0_0, 0_8, 2_5 \rangle$	$\langle 1_1, 0_1, 3_7, 3_8 \rangle$	$\langle 3_3, 0_2, 3_9, 1_7 \rangle$	$\langle 1_0, 1_2, 1_9, 0_6 \rangle$	$\langle 0_1, 3_2, 0_9, 0_6 \rangle$	$\langle 0_3, 0_0, 2_9, 1_5 \rangle$	$\langle 2_0, 1_2, 0_7, 1_8 \rangle$
$\langle 3_3, 2_3, 0_8, 1_4 \rangle$	$\langle 2_3, 1_1, 0_6, 1_8 \rangle$	$\langle 1_2, 2_2, 0_8, 0_4 \rangle$	$\langle 0_3, 2_0, 0_8, 1_9 \rangle$	$\langle 3_0, 3_1, 0_7, 0_9 \rangle$	$\langle 2_2, 1_1, 3_4, 0_9 \rangle$	$\langle 0_1, 3_3, 3_5, 0_7 \rangle$	

$n = 33$ :

$\langle \infty, 2_0, 0_8, 0_9 \rangle$	$\langle \infty, 2_1, 3_4, 3_6 \rangle$	$\langle \infty, 3_3, 2_{11}, 3_5 \rangle$	$\langle \infty, 3_2, 1_{10}, 3_7 \rangle$	$\langle 2_2, 3_3, 0_5, a_1 \rangle$	$\langle 3_1, 1_0, 0_7, a_1 \rangle$	$\langle 2_1, 1_3, 1_{11}, 3_8 \rangle$	$\langle 2_2, 1_2, 1_{11}, 1_5 \rangle$
$\langle 3_1, 0_1, 3_9, 2_7 \rangle$	$\langle 1_2, 1_1, 0_4, 2_{10} \rangle$	$\langle 0_1, 1_2, 2_4, 3_8 \rangle$	$\langle 0_3, 0_0, 3_9, 1_7 \rangle$	$\langle 1_3, 3_2, 2_6, 2_9 \rangle$	$\langle 3_0, 0_0, 3_6, 1_{11} \rangle$	$\langle 2_2, 1_0, 0_4, 1_{10} \rangle$	$\langle 3_0, 1_2, 3_{11}, 0_8 \rangle$
$\langle 1_0, 0_2, 1_9, 1_5 \rangle$	$\langle 1_0, 1_1, 2_5, 3_6 \rangle$	$\langle 1_0, 2_1, 0_8, 0_{11} \rangle$	$\langle 0_1, 2_2, 3_6, 2_{10} \rangle$	$\langle 0_2, 0_3, 2_9, 3_7 \rangle$	$\langle 3_0, 1_3, 2_{10}, 0_6 \rangle$	$\langle 2_0, 2_2, 2_8, 0_7 \rangle$	$\langle 3_1, 3_3, 3_6, 0_{11} \rangle^s$
$\langle 0_3, 1_3, 1_8, 3_4 \rangle$	$\langle 0_0, 3_1, 1_9, 2_{10} \rangle$	$\langle 0_0, 3_3, 1_{10}, 2_5 \rangle$	$\langle 3_2, 0_1, 0_7, 0_8 \rangle$	$\langle 2_2, 1_3, 3_{11}, 2_4 \rangle$	$\langle 0_1, 1_3, 0_{10}, 1_9 \rangle$	$\langle 0_0, 1_3, 1_4, 3_5 \rangle$	$\langle 0_0, 2_0, 0_4, 2_4 \rangle^s$
$\langle 0_1, 2_1, 0_5, 2_5 \rangle^s$	$\langle 0_2, 2_2, 0_6, 2_6 \rangle^s$	$\langle 0_3, 2_3, 0_7, 2_7 \rangle^s$					

Note that each of the codewords marked  $s$  only generates two codewords. ■

**Lemma 8.3:**  $T(2, 21; 2, n; 6) = 5n$  for each odd  $n$  and  $21 \leq n \leq 41$ .

*Proof:* Let  $X_1 = (\mathbb{Z}_5 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $21 \leq n \leq 29$ , let  $X_2 = (\mathbb{Z}_5 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-20\})$ ; for  $31 \leq n \leq 39$ , let  $X_2 = (\mathbb{Z}_5 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-30\})$ ; for  $n = 41$ , let  $X_2 = (\mathbb{Z}_5 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ . Denote  $X = X_1 \cup X_2$ . The desired codes of size  $5n$  are constructed on  $\mathbb{Z}_2^X$  and the base codewords are listed as follows.

$n = 21$ :

$\langle \infty, 2_0, 4_5, 4_4 \rangle$	$\langle \infty, 0_1, 0_7, 3_6 \rangle$	$\langle 1_1, 2_0, 2_4, a_1 \rangle$	$\langle 2_2, 0_3, 3_6, a_1 \rangle$	$\langle 2_2, 2_3, 3_5, 3_7 \rangle$	$\langle 4_1, 4_2, 1_4, 3_5 \rangle$	$\langle 1_2, 4_2, 4_6, 1_5 \rangle$	$\langle 1_2, 4_0, 4_5, 0_4 \rangle$
$\langle 1_3, 4_3, 4_5, 3_6 \rangle$	$\langle 1_2, 0_3, 0_6, 1_4 \rangle$	$\langle 0_3, 4_2, 2_7, 1_6 \rangle$	$\langle 2_0, 0_1, 3_7, 2_7 \rangle$	$\langle 0_1, 4_1, 4_6, 4_4 \rangle$	$\langle 0_3, 4_3, 2_4, 4_4 \rangle$	$\langle 1_0, 2_0, 0_6, 2_6 \rangle$	$\langle 1_0, 3_1, 4_5, 0_5 \rangle$
$\langle 0_0, 4_3, 1_5, 3_7 \rangle$	$\langle 3_0, 4_1, 2_4, 0_6 \rangle$	$\langle 0_1, 3_1, 4_7, 3_5 \rangle$	$\langle 2_0, 2_2, 0_4, 1_7 \rangle$	$\langle 0_2, 2_3, 0_7, 2_7 \rangle$			

$n = 23$ :

$\langle \infty, 3_0, 3_4, 3_5 \rangle$	$\langle \infty, 2_1, 2_6, 2_7 \rangle$	$\langle 2_0, 2_1, 3_4, a_1 \rangle$	$\langle 1_2, 3_3, 3_5, a_1 \rangle$	$\langle 2_0, 3_1, 0_4, a_2 \rangle$	$\langle 1_2, 2_3, 4_6, a_2 \rangle$	$\langle 0_2, 4_3, 3_5, a_3 \rangle$	$\langle 3_0, 0_1, 0_4, a_3 \rangle$
$\langle 1_2, 1_3, 1_6, 3_7 \rangle$	$\langle 0_0, 4_1, 3_7, 2_7 \rangle$	$\langle 0_1, 1_2, 1_7, 2_6 \rangle$	$\langle 0_3, 1_3, 0_4, 0_7 \rangle$	$\langle 1_2, 4_3, 0_5, 0_7 \rangle$	$\langle 1_2, 0_2, 2_4, 1_5 \rangle$	$\langle 0_1, 2_1, 3_5, 3_6 \rangle$	$\langle 0_0, 4_0, 1_5, 3_5 \rangle$
$\langle 0_1, 1_1, 0_5, 4_4 \rangle$	$\langle 0_0, 2_0, 1_6, 0_6 \rangle$	$\langle 0_2, 3_2, 3_4, 2_6 \rangle$	$\langle 0_3, 2_3, 3_4, 3_6 \rangle$	$\langle 0_0, 3_2, 4_7, 1_7 \rangle$	$\langle 1_0, 3_3, 0_4, 1_7 \rangle$	$\langle 0_1, 0_3, 2_5, 4_6 \rangle$	

$n = 25$ :

$\langle \infty, 2_1, 2_7, 0_6 \rangle$	$\langle \infty, 3_3, 2_5, 4_4 \rangle$	$\langle 3_3, 2_2, 1_4, a_1 \rangle$	$\langle 3_1, 0_0, 4_7, a_1 \rangle$	$\langle 0_0, 0_1, 3_4, a_2 \rangle$	$\langle 3_2, 0_3, 0_5, a_2 \rangle$	$\langle 1_2, 1_0, 1_7, a_3 \rangle$	$\langle 3_3, 1_1, 2_4, a_3 \rangle$
$\langle 3_0, 2_1, 3_5, a_4 \rangle$	$\langle 3_3, 4_2, 0_4, a_4 \rangle$	$\langle 1_1, 4_3, 0_7, a_5 \rangle$	$\langle 1_2, 2_0, 1_6, a_5 \rangle$	$\langle 2_1, 3_2, 1_5, 1_4 \rangle$	$\langle 3_3, 3_2, 2_6, 2_7 \rangle$	$\langle 1_1, 3_2, 3_4, 4_5 \rangle$	$\langle 1_0, 4_0, 0_5, 3_4 \rangle$
$\langle 3_0, 3_3, 3_6, 3_4 \rangle$	$\langle 4_2, 3_2, 1_6, 0_7 \rangle$	$\langle 1_3, 1_1, 2_6, 3_5 \rangle$	$\langle 4_0, 1_1, 0_6, 1_6 \rangle$	$\langle 1_1, 0_0, 3_6, 1_4 \rangle$	$\langle 1_2, 1_1, 1_5, 4_7 \rangle$	$\langle 0_3, 2_2, 1_5, 3_6 \rangle$	$\langle 2_0, 1_3, 4_5, 4_7 \rangle$
$\langle 0_0, 1_3, 1_7, 3_7 \rangle$							

$n = 27$ :

$\langle \infty, 0_2, 0_5, 0_4 \rangle$	$\langle \infty, 0_1, 3_7, 0_6 \rangle$	$\langle 3_3, 3_2, 4_6, a_1 \rangle$	$\langle 2_1, 2_0, 0_4, a_1 \rangle$	$\langle 0_3, 2_2, 3_7, a_2 \rangle$	$\langle 2_0, 0_1, 4_4, a_2 \rangle$	$\langle 3_1, 1_0, 4_5, a_3 \rangle$	$\langle 0_2, 2_3, 2_6, a_3 \rangle$
$\langle 4_0, 3_1, 2_6, a_4 \rangle$	$\langle 0_2, 4_3, 4_7, a_4 \rangle$	$\langle 4_3, 3_0, 2_4, a_5 \rangle$	$\langle 1_1, 3_2, 3_6, a_5 \rangle$	$\langle 1_3, 2_1, 2_7, a_6 \rangle$	$\langle 2_2, 0_0, 0_4, a_6 \rangle$	$\langle 1_0, 4_2, 2_5, a_7 \rangle$	$\langle 0_1, 0_3, 2_7, a_7 \rangle$
$\langle 1_3, 2_0, 2_5, 4_5 \rangle$	$\langle 0_1, 1_3, 3_6, 1_4 \rangle$	$\langle 4_1, 1_1, 1_5, 0_7 \rangle$	$\langle 4_2, 0_3, 3_6, 4_7 \rangle$	$\langle 3_0, 4_0, 0_7, 3_6 \rangle$	$\langle 3_2, 0_2, 4_5, 2_4 \rangle$	$\langle 0_0, 1_1, 2_6, 1_4 \rangle$	$\langle 2_3, 1_3, 1_5, 3_4 \rangle$
$\langle 2_2, 1_1, 3_4, 4_5 \rangle$	$\langle 0_2, 4_0, 2_7, 3_7 \rangle$	$\langle 0_0, 2_3, 4_5, 1_6 \rangle$					

$n = 29$ :

$\langle \infty, 0_0, 2_5, 1_4 \rangle$	$\langle \infty, 0_1, 4_7, 0_6 \rangle$	$\langle 4_1, 0_2, 1_7, a_1 \rangle$	$\langle 3_0, 0_3, 0_4, a_1 \rangle$	$\langle 2_3, 2_2, 1_5, a_2 \rangle$	$\langle 4_0, 0_1, 3_4, a_2 \rangle$	$\langle 0_3, 4_2, 2_5, a_3 \rangle$	$\langle 4_1, 2_0, 0_6, a_3 \rangle$
$\langle 1_1, 3_0, 4_5, a_4 \rangle$	$\langle 1_3, 3_2, 0_4, a_4 \rangle$	$\langle 0_0, 4_1, 3_5, a_5 \rangle$	$\langle 1_3, 2_2, 0_6, a_5 \rangle$	$\langle 0_0, 0_2, 2_7, a_6 \rangle$	$\langle 4_1, 3_3, 4_4, a_6 \rangle$	$\langle 1_3, 4_1, 3_6, a_7 \rangle$	$\langle 1_2, 0_0, 0_7, a_7 \rangle$
$\langle 2_0, 0_3, 2_4, a_8 \rangle$	$\langle 3_2, 4_1, 4_5, a_8 \rangle$	$\langle 2_1, 2_3, 2_7, a_9 \rangle$	$\langle 2_0, 1_2, 1_5, a_9 \rangle$	$\langle 3_0, 0_0, 4_6, 4_7 \rangle$	$\langle 2_3, 3_3, 4_7, 3_5 \rangle$	$\langle 1_2, 1_1, 0_4, 3_5 \rangle$	$\langle 0_3, 0_0, 3_4, 3_7 \rangle$
$\langle 2_1, 0_3, 0_6, 3_5 \rangle$	$\langle 3_3, 4_0, 4_6, 1_6 \rangle$	$\langle 1_2, 4_2, 2_4, 0_6 \rangle$	$\langle 2_1, 0_2, 3_7, 0_7 \rangle$	$\langle 0_1, 2_2, 2_4, 2_6 \rangle$			

$n = 31$ :

$\langle \infty, 3_1, 3_6, 3_7 \rangle$	$\langle \infty, 0_0, 2_4, 4_5 \rangle$	$\langle \infty, 1_2, 0_8, 1_9 \rangle$	$\langle 4_0, 4_1, 0_4, a_1 \rangle$	$\langle 1_2, 0_3, 0_5, a_1 \rangle$	$\langle 1_0, 1_3, 4_8, 2_9 \rangle$	$\langle 1_2, 2_3, 3_6, 3_5 \rangle$	$\langle 0_0, 1_0, 4_9, 2_8 \rangle$
$\langle 0_2, 3_3, 3_8, a_3 \rangle$	$\langle 0_1, 3_1, 2_5, 3_8 \rangle$	$\langle 0_2, 2_2, 0_6, 3_4 \rangle$	$\langle 0_0, 2_3, 0_5, 4_7 \rangle$	$\langle 0_1, 4_1, 4_4, 3_6 \rangle$	$\langle 1_1, 1_2, 0_9, 3_4 \rangle$	$\langle 0_1, 3_2, 3_5, 0_9 \rangle$	$\langle 2_0, 3_1, 3_5, 4_9 \rangle$
$\langle 1_0, 0_1, 4_7, 1_8 \rangle$	$\langle 0_2, 1_2, 0_4, 1_7 \rangle$	$\langle 1_1, 2_3, 2_7, 3_8 \rangle$	$\langle 1_1, 3_3, 2_6, 0_8 \rangle$	$\langle 0_0, 4_2, 2_5, 2_7 \rangle$	$\langle 0_3, 1_3, 3_6, 2_4 \rangle$	$\langle 0_3, 3_3, 1_7, 2_9 \rangle$	$\langle 1_2, 3_3, 2_5, 2_8 \rangle$
$\langle 1_1, 1_3, 4_9, 4_4 \rangle$	$\langle 1_0, 4_3, 4_4, 4_6 \rangle$	$\langle 0_0, 2_2, 1_7, 4_8 \rangle$	$\langle 0_0, 3_0, 0_6, 4_6 \rangle$	$\langle 1_0, 2_3, 1_4, 4_5 \rangle$	$\langle 0_1, 1_2, 3_7, 2_6 \rangle$	$\langle 0_0, 3_1, 0_7, 0_9 \rangle$	

$n = 33$ :

$\langle \infty, 4_2, 1_8, 4_9 \rangle$	$\langle \infty, 3_0, 3_4, 3_5 \rangle$	$\langle \infty, 1_1, 1_6, 3_7 \rangle$	$\langle 3_1, 3_0, 4_4, a_1 \rangle$	$\langle 3_3, 3_2, 3_5, a_1 \rangle$	$\langle 1_3, 0_2, 2_5, a_2 \rangle$	$\langle 3_0, 4_1, 1_4, a_2 \rangle$	$\langle 3_2, 2_3, 1_5, a_3 \rangle$
$\langle 3_0, 0_1, 4_7, a_3 \rangle$	$\langle 1_3, 2_3, 3_7, 1_4 \rangle$	$\langle 2_1, 4_3, 2_7, 3_6 \rangle$	$\langle 0_0, 1_3, 0_9, 3_9 \rangle$	$\langle 0_2, 2_2, 3_4, 2_4 \rangle$	$\langle 1_0, 2_2, 1_7, 0_8 \rangle$	$\langle 1_1, 3_1, 2_9, 0_8 \rangle$	$\langle 0_3, 3_3, 4_8, 1_4 \rangle$
$\langle 1_0, 4_2, 0_9, 3_5 \rangle$	$\langle 2_1, 0_2, 0_6, 4_9 \rangle$	$\langle 1_1, 2_3, 4_8, 4_5 \rangle$	$\langle 1_1, 1_2, 0_8, 3_6 \rangle$	$\langle 1_2, 3_3, 1_8, 4_6 \rangle$	$\langle 4_1, 0_2, 0_7, 1_5 \rangle$	$\langle 0_0, 2_2, 4_7, 3_7 \rangle$	$\langle 2_0, 4_0, 1_6, 0_6 \rangle$
$\langle 1_1, 0_3, 1_9, 0_6 \rangle$	$\langle 1_0, 4_3, 2_5, 2_9 \rangle$	$\langle 0_1, 4_1, 0_5, 3_4 \rangle$	$\langle 0_0, 0_3, 2_6, 0_8 \rangle$	$\langle 4_0, 3_2, 2_8, 1_7 \rangle$	$\langle 2_2, 0_3, 0_9, 3_6 \rangle$	$\langle 1_0, 0_0, 4_5, 2_8 \rangle$	$\langle 0_0, 0_2, 2_9, 4_4 \rangle$
$\langle 0_1, 3_3, 0_4, 3_7 \rangle$							

$n = 35$ :

$\langle \infty, 0_3, 3_7, 2_9 \rangle$	$\langle \infty, 0_0, 0_5, 1_4 \rangle$	$\langle \infty, 0_2, 0_8, 4_6 \rangle$	$\langle 4_1, 4_0, 0_7, a_1 \rangle$	$\langle 3_3, 0_2, 3_5, a_1 \rangle$	$\langle 2_0, 0_1, 2_4, a_2 \rangle$	$\langle 4_3, 3_2, 4_6, a_2 \rangle$	$\langle 2_3, 0_2, 4_7, a_3 \rangle$
$\langle 1_1, 2_0, 1_5, a_3 \rangle$	$\langle 0_3, 3_0, 4_6, a_4 \rangle$	$\langle 1_1, 1_2, 4_7, a_4 \rangle$	$\langle 3_1, 1_0, 0_7, a_5 \rangle$	$\langle 1_2, 0_3, 1_4, a_5 \rangle$	$\langle 3_2, 2_2, 4_9, 3_5 \rangle$	$\langle 4_3, 4_2, 1_6, 2_9 \rangle$	$\langle 1_3, 0_1, 1_9, 2_9 \rangle$
$\langle 1_1, 4_3, 1_4, 4_4 \rangle$	$\langle 0_1, 4_3, 0_7, 0_6 \rangle$	$\langle 0_1, 2_2, 1_4, 4_5 \rangle$	$\langle 2_0, 0_3, 4_9, 4_7 \rangle$	$\langle 3_2, 2_0, 0_7, 3_9 \rangle$	$\langle 1_3, 1_1, 0_4, 0_8 \rangle$	$\langle 4_1, 2_2, 3_7, 4_8 \rangle$	$\langle 2_0, 4_0, 2_9, 0_5 \rangle$
$\langle 4_2, 2_0, 1_4, 0_4 \rangle$	$\langle 4_1, 1_1, 0_6, 2_8 \rangle$	$\langle 0_0, 1_1, 3_6, 4_6 \rangle$	$\langle 2_2, 4_0, 3_8, 1_8 \rangle$	$\langle 3_1, 4_1, 0_5, 3_9 \rangle$	$\langle 2_2, 2_0, 1_9, 2_6 \rangle$	$\langle 4_2, 0_0, 2_4, 2_6 \rangle$	$\langle 0_0, 0_3, 0_7, 3_8 \rangle$
$\langle 2_3, 4_3, 4_8, 3_5 \rangle$	$\langle 0_0, 4_3, 2_5, 0_8 \rangle$	$\langle 0_1, 4_2, 3_5, 2_8 \rangle$					

$n = 37$ :

$\langle \infty, 3_0, 2_5, 2_4 \rangle$	$\langle \infty, 3_3, 0_8, 3_9 \rangle$	$\langle \infty, 0_2, 2_6, 1_7 \rangle$	$\langle 3_1, 4_2, 4_7, a_1 \rangle$	$\langle 2_3, 2_0, 2_5, a_1 \rangle$	$\langle 2_1, 0_2, 4_4, a_2 \rangle$	$\langle 3_3, 4_0, 1_6, a_2 \rangle$	$\langle 0_2, 3_3, 2_4, a_3 \rangle$
$\langle 0_0, 3_1, 0_6, a_3 \rangle$	$\langle 1_0, 4_3, 4_4, a_4 \rangle$	$\langle 4_1, 4_2, 4_5, a_4 \rangle$	$\langle 1_3, 1_2, 0_6, a_5 \rangle$	$\langle 0_1, 0_0, 3_5, a_5 \rangle$	$\langle 0_2, 2_0, 3_4, a_6 \rangle$	$\langle 0_1, 0_3, 4_5, a_6 \rangle$	$\langle 3_2, 1_0, 0_7, a_7 \rangle$
$\langle 4_1, 3_3, 0_4, a_7 \rangle$	$\langle 3_1, 1_1, 0_9, 1_4 \rangle$	$\langle 4_3, 3_2, 4_6, 1_7 \rangle$	$\langle 0_2, 1_1, 0_4, 2_8 \rangle$	$\langle 0_0, 4_2, 3_8, 4_6 \rangle$	$\langle 2_3, 1_3, 0_7, 4_5 \rangle$	$\langle 0_0, 2_0, 4_9, 0_7 \rangle$	$\langle 2_0, 4_3, 3_9, 4_8 \rangle$
$\langle 3_0, 4_3, 1_9, 0_7 \rangle$	$\langle 3_1, 0_3, 1_6, 1_9 \rangle$	$\langle 0_0, 4_1, 4_8, 1_5 \rangle$	$\langle 1_0, 2_1, 2_7, 1_8 \rangle$	$\langle 2_1, 4_2, 3_5, 0_8 \rangle$	$\langle 0_1, 3_0, 1_6, 4_6 \rangle$	$\langle 0_0, 1_2, 2_4, 1_8 \rangle$	$\langle 1_1, 2_1, 0_7, 2_9 \rangle$
$\langle 4_2, 3_3, 3_7, 1_9 \rangle$	$\langle 2_1, 0_3, 4_8, 2_6 \rangle$	$\langle 0_2, 0_0, 2_5, 0_9 \rangle$	$\langle 3_3, 0_3, 1_4, 1_8 \rangle$	$\langle 0_2, 2_2, 3_5, 3_9 \rangle$			

$n = 39$ :

$\langle \infty, 31, 49, 48 \rangle$	$\langle \infty, 02, 14, 15 \rangle$	$\langle \infty, 23, 36, 37 \rangle$	$\langle 22, 13, 15, a_1 \rangle$	$\langle 20, 31, 17, a_1 \rangle$	$\langle 33, 20, 36, a_2 \rangle$	$\langle 31, 22, 14, a_2 \rangle$	$\langle 00, 21, 24, a_3 \rangle$
$\langle 22, 43, 05, a_3 \rangle$	$\langle 30, 31, 35, a_4 \rangle$	$\langle 23, 42, 14, a_4 \rangle$	$\langle 13, 40, 05, a_5 \rangle$	$\langle 22, 41, 36, a_5 \rangle$	$\langle 12, 20, 37, a_6 \rangle$	$\langle 43, 11, 04, a_6 \rangle$	$\langle 20, 13, 34, a_7 \rangle$
$\langle 21, 42, 37, a_7 \rangle$	$\langle 22, 40, 26, a_8 \rangle$	$\langle 43, 41, 25, a_8 \rangle$	$\langle 30, 02, 36, a_9 \rangle$	$\langle 21, 33, 27, a_9 \rangle$	$\langle 41, 21, 15, a_8 \rangle$	$\langle 01, 41, 06, 29 \rangle$	$\langle 40, 00, 44, 09 \rangle$
$\langle 43, 23, 48, 47 \rangle$	$\langle 21, 13, 19, 47 \rangle$	$\langle 13, 02, 09, 28 \rangle$	$\langle 03, 20, 38, 04 \rangle$	$\langle 00, 31, 45, 27 \rangle$	$\langle 40, 43, 29, 15 \rangle$	$\langle 32, 42, 48, 09 \rangle$	$\langle 10, 30, 09, 18 \rangle$
$\langle 00, 41, 26, 28 \rangle$	$\langle 03, 13, 29, 36 \rangle$	$\langle 11, 12, 08, 36 \rangle$	$\langle 40, 42, 27, 47 \rangle$	$\langle 20, 32, 05, 18 \rangle$	$\langle 42, 31, 44, 39 \rangle$	$\langle 02, 03, 34, 46 \rangle$	

$n = 41$ :

$\langle \infty, 10, 310, 36 \rangle$	$\langle \infty, 03, 111, 35 \rangle$	$\langle \infty, 02, 34, 47 \rangle$	$\langle \infty, 31, 08, 39 \rangle$	$\langle 01, 30, 07, a_1 \rangle$	$\langle 32, 33, 36, a_1 \rangle$	$\langle 21, 40, 36, 45 \rangle$	$\langle 01, 11, 48, 210 \rangle$
$\langle 21, 32, 49, 19 \rangle$	$\langle 11, 31, 07, 311 \rangle$	$\langle 22, 03, 16, 18 \rangle$	$\langle 02, 23, 08, 410 \rangle$	$\langle 32, 31, 44, 48 \rangle$	$\langle 40, 12, 411, 34 \rangle$	$\langle 01, 13, 06, 39 \rangle$	$\langle 13, 20, 37, 05 \rangle$
$\langle 00, 11, 110, 25 \rangle$	$\langle 23, 33, 19, 28 \rangle$	$\langle 00, 33, 06, 49 \rangle$	$\langle 21, 02, 44, 010 \rangle$	$\langle 40, 13, 47, 27 \rangle$	$\langle 20, 02, 110, 36 \rangle$	$\langle 03, 21, 110, 05 \rangle$	$\langle 00, 03, 09, 36 \rangle$
$\langle 01, 23, 111, 44 \rangle$	$\langle 20, 40, 48, 04 \rangle$	$\langle 23, 21, 24, 011 \rangle$	$\langle 00, 10, 311, 39 \rangle$	$\langle 13, 22, 25, 311 \rangle$	$\langle 31, 40, 14, 38 \rangle$	$\langle 11, 02, 27, 15 \rangle$	$\langle 20, 22, 111, 010 \rangle$
$\langle 02, 12, 45, 09 \rangle$	$\langle 01, 22, 46, 36 \rangle$	$\langle 11, 10, 05, 29 \rangle$	$\langle 02, 32, 011, 37 \rangle$	$\langle 31, 23, 17, 211 \rangle$	$\langle 02, 10, 28, 25 \rangle$	$\langle 12, 23, 27, 48 \rangle$	$\langle 23, 03, 34, 010 \rangle$
$\langle 00, 13, 04, 010 \rangle$							

**Lemma 8.4:**  $T(2, 25; 2, n; 6) = 6n$  for each odd  $n$  and  $25 \leq n \leq 49$ . ■

*Proof:* Let  $X_1 = (\mathbb{Z}_6 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $25 \leq n \leq 35$ , let  $X_2 = (\mathbb{Z}_6 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-24\})$ ; for  $37 \leq n \leq 47$ , let  $X_2 = (\mathbb{Z}_6 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-36\})$ ; for  $n = 49$ , let  $X_2 = (\mathbb{Z}_6 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ . Denote  $X = X_1 \cup X_2$ . The desired codes of size  $6n$  are constructed on  $\mathbb{Z}_2^X$  and the base codewords are listed as follows.

$n = 25$ :

$\langle \infty, 50, 05, 54 \rangle$	$\langle \infty, 51, 57, 56 \rangle$	$\langle 12, 33, 15, a_1 \rangle$	$\langle 10, 11, 24, a_1 \rangle$	$\langle 50, 33, 46, 57 \rangle$	$\langle 00, 50, 44, 24 \rangle$	$\langle 03, 30, 35, 36 \rangle$	$\langle 50, 42, 47, 25 \rangle$
$\langle 02, 00, 27, 55 \rangle$	$\langle 31, 42, 14, 06 \rangle$	$\langle 40, 01, 56, 16 \rangle$	$\langle 52, 53, 44, 05 \rangle$	$\langle 40, 03, 05, 26 \rangle$	$\langle 41, 53, 57, 34 \rangle$	$\langle 40, 21, 25, 06 \rangle$	$\langle 01, 10, 47, 27 \rangle$
$\langle 21, 31, 05, 45 \rangle$	$\langle 13, 02, 14, 16 \rangle$	$\langle 33, 43, 54, 27 \rangle$	$\langle 32, 52, 36, 54 \rangle$	$\langle 03, 21, 46, 17 \rangle$	$\langle 32, 51, 14, 27 \rangle$	$\langle 43, 11, 14, 05 \rangle$	$\langle 52, 33, 26, 07 \rangle$
$\langle 02, 12, 35, 47 \rangle$							

$n = 27$ :

$\langle \infty, 11, 17, 16 \rangle$	$\langle \infty, 20, 24, 25 \rangle$	$\langle 12, 13, 15, a_1 \rangle$	$\langle 20, 21, 34, a_1 \rangle$	$\langle 00, 11, 34, a_2 \rangle$	$\langle 22, 13, 05, a_2 \rangle$	$\langle 50, 11, 14, a_3 \rangle$	$\langle 12, 23, 26, a_3 \rangle$
$\langle 00, 23, 46, 35 \rangle$	$\langle 00, 43, 26, 07 \rangle$	$\langle 01, 51, 34, 55 \rangle$	$\langle 00, 02, 27, 17 \rangle$	$\langle 02, 43, 57, 15 \rangle$	$\langle 03, 13, 04, 57 \rangle$	$\langle 11, 12, 45, 04 \rangle$	$\langle 10, 51, 06, 46 \rangle$
$\langle 01, 21, 45, 47 \rangle$	$\langle 03, 23, 44, 56 \rangle$	$\langle 01, 22, 26, 57 \rangle$	$\langle 10, 02, 47, 07 \rangle$	$\langle 02, 22, 56, 46 \rangle$	$\langle 02, 33, 44, 55 \rangle$	$\langle 02, 52, 04, 24 \rangle$	$\langle 00, 10, 25, 54 \rangle$
$\langle 00, 13, 55, 47 \rangle$	$\langle 00, 31, 06, 45 \rangle$	$\langle 01, 33, 46, 37 \rangle$					

$n = 29$ :

$\langle \infty, 41, 47, 06 \rangle$	$\langle \infty, 10, 05, 14 \rangle$	$\langle 41, 40, 54, a_1 \rangle$	$\langle 13, 02, 56, a_1 \rangle$	$\langle 33, 52, 24, a_2 \rangle$	$\langle 00, 21, 57, a_2 \rangle$	$\langle 41, 10, 04, a_3 \rangle$	$\langle 23, 32, 45, a_3 \rangle$
$\langle 21, 30, 36, a_4 \rangle$	$\langle 02, 33, 54, a_4 \rangle$	$\langle 43, 22, 47, a_5 \rangle$	$\langle 10, 51, 56, a_5 \rangle$	$\langle 30, 03, 04, 37 \rangle$	$\langle 23, 33, 36, 35 \rangle$	$\langle 41, 02, 07, 24 \rangle$	$\langle 40, 32, 55, 15 \rangle$
$\langle 23, 43, 37, 16 \rangle$	$\langle 12, 22, 46, 15 \rangle$	$\langle 40, 02, 57, 17 \rangle$	$\langle 31, 02, 04, 06 \rangle$	$\langle 21, 32, 17, 07 \rangle$	$\langle 51, 23, 07, 15 \rangle$	$\langle 21, 02, 35, 14 \rangle$	$\langle 50, 53, 17, 16 \rangle$
$\langle 51, 01, 55, 46 \rangle$	$\langle 40, 00, 16, 05 \rangle$	$\langle 22, 40, 04, 36 \rangle$	$\langle 41, 33, 44, 15 \rangle$	$\langle 00, 13, 44, 45 \rangle$			

$n = 31$ :

$\langle \infty, 21, 27, 06 \rangle$	$\langle \infty, 50, 54, 35 \rangle$	$\langle 11, 10, 57, a_1 \rangle$	$\langle 12, 13, 26, a_1 \rangle$	$\langle 02, 53, 36, a_2 \rangle$	$\langle 51, 40, 14, a_2 \rangle$	$\langle 11, 50, 14, a_3 \rangle$	$\langle 02, 33, 15, a_3 \rangle$
$\langle 21, 50, 05, a_4 \rangle$	$\langle 53, 32, 56, a_4 \rangle$	$\langle 30, 11, 16, a_5 \rangle$	$\langle 43, 02, 04, a_5 \rangle$	$\langle 02, 13, 46, a_6 \rangle$	$\langle 30, 21, 14, a_6 \rangle$	$\langle 23, 41, 16, a_7 \rangle$	$\langle 50, 12, 47, a_7 \rangle$
$\langle 13, 11, 05, 15 \rangle$	$\langle 02, 41, 55, 06 \rangle$	$\langle 23, 13, 24, 35 \rangle$	$\langle 31, 22, 47, 55 \rangle$	$\langle 11, 03, 07, 47 \rangle$	$\langle 12, 02, 54, 17 \rangle$	$\langle 13, 20, 27, 37 \rangle$	$\langle 31, 02, 57, 14 \rangle$
$\langle 00, 32, 26, 35 \rangle$	$\langle 23, 03, 54, 57 \rangle$	$\langle 40, 22, 07, 05 \rangle$	$\langle 30, 40, 35, 36 \rangle$	$\langle 21, 32, 55, 54 \rangle$	$\langle 01, 33, 14, 56 \rangle$	$\langle 00, 20, 14, 36 \rangle$	

$n = 33$ :

$\langle \infty, 11, 46, 17 \rangle$	$\langle \infty, 40, 44, 45 \rangle$	$\langle 21, 20, 34, a_1 \rangle$	$\langle 33, 32, 55, a_1 \rangle$	$\langle 03, 52, 35, a_2 \rangle$	$\langle 21, 10, 44, a_2 \rangle$	$\langle 52, 13, 05, a_3 \rangle$	$\langle 11, 50, 14, a_3 \rangle$
$\langle 21, 50, 05, a_4 \rangle$	$\langle 52, 23, 54, a_4 \rangle$	$\langle 01, 20, 46, a_5 \rangle$	$\langle 02, 43, 47, a_5 \rangle$	$\langle 52, 43, 25, a_6 \rangle$	$\langle 41, 50, 34, a_6 \rangle$	$\langle 10, 42, 45, a_7 \rangle$	$\langle 03, 11, 37, a_7 \rangle$
$\langle 40, 42, 35, a_8 \rangle$	$\langle 51, 13, 07, a_8 \rangle$	$\langle 11, 53, 05, a_9 \rangle$	$\langle 50, 42, 07, a_9 \rangle$	$\langle 03, 13, 54, 14 \rangle$	$\langle 20, 02, 57, 17 \rangle$	$\langle 01, 42, 56, 47 \rangle$	$\langle 02, 42, 36, 34 \rangle$
$\langle 30, 42, 46, 06 \rangle$	$\langle 21, 53, 17, 55 \rangle$	$\langle 03, 23, 56, 46 \rangle$	$\langle 10, 50, 37, 35 \rangle$	$\langle 20, 53, 14, 06 \rangle$	$\langle 20, 13, 16, 27 \rangle$	$\langle 31, 32, 07, 14 \rangle$	$\langle 11, 01, 26, 15 \rangle$
$\langle 01, 22, 34, 06 \rangle$							

$n = 35$ :

$\langle \infty, 33, 36, 47 \rangle$	$\langle \infty, 10, 55, 24 \rangle$	$\langle 13, 02, 17, a_1 \rangle$	$\langle 31, 50, 55, a_1 \rangle$	$\langle 11, 00, 44, a_2 \rangle$	$\langle 53, 22, 25, a_2 \rangle$	$\langle 12, 33, 54, a_3 \rangle$	$\langle 40, 01, 16, a_3 \rangle$
$\langle 22, 11, 56, a_4 \rangle$	$\langle 53, 20, 27, a_4 \rangle$	$\langle 30, 01, 55, a_5 \rangle$	$\langle 13, 32, 56, a_5 \rangle$	$\langle 02, 53, 15, a_6 \rangle$	$\langle 40, 31, 06, a_6 \rangle$	$\langle 00, 02, 46, a_7 \rangle$	$\langle 21, 43, 25, a_7 \rangle$
$\langle 33, 31, 24, a_8 \rangle$	$\langle 32, 40, 07, a_8 \rangle$	$\langle 23, 51, 17, a_9 \rangle$	$\langle 32, 20, 15, a_9 \rangle$	$\langle 52, 10, 47, a_{10} \rangle$	$\langle 23, 31, 36, a_{10} \rangle$	$\langle 22, 23, 16, a_{11} \rangle$	$\langle 50, 51, 07, a_{11} \rangle$
$\langle 42, 20, 54, 14 \rangle$	$\langle 22, 51, 14, 47 \rangle$	$\langle 00, 20, 16, 24 \rangle$	$\langle 32, 41, 17, 25 \rangle$	$\langle 32, 11, 36, 54 \rangle$	$\langle 13, 00, 57, 15 \rangle$	$\langle 53, 43, 16, 54 \rangle$	$\langle 42, 41, 44, 47 \rangle$
$\langle 30, 53, 17, 05 \rangle$	$\langle 51, 32, 46, 05 \rangle$	$\langle 01, 43, 14, 35 \rangle$					

$n = 37$ :

$\langle \infty, 02, 09, 18 \rangle$	$\langle \infty, 31, 47, 36 \rangle$	$\langle \infty, 40, 45, 24 \rangle$	$\langle 01, 00, 14, a_1 \rangle$	$\langle 03, 42, 36, a_1 \rangle$	$\langle 32, 52, 45, 04 \rangle$	$\langle 12, 41, 04, 09 \rangle$	$\langle 00, 10, 04, 39 \rangle$
$\langle 13, 01, 19, 35 \rangle$	$\langle 01, 50, 57, 58 \rangle$	$\langle 03, 40, 29, 18 \rangle$	$\langle 52, 53, 58, 34 \rangle$	$\langle 23, 31, 07, 17 \rangle$	$\langle 11, 30, 56, 15 \rangle$	$\langle 03, 43, 28, 34 \rangle$	$\langle 02, 53, 16, 29 \rangle$
$\langle 00, 31, 06, 56 \rangle$	$\langle 21, 31, 08, 04 \rangle$	$\langle 51, 33, 08, 54 \rangle$	$\langle 01, 52, 09, 54 \rangle$	$\langle 42, 32, 08, 16 \rangle$	$\langle 51, 02, 17, 58 \rangle$	$\langle 11, 43, 26, 38 \rangle$	$\langle 43, 02, 47, 07 \rangle$
$\langle 11, 50, 09, 59 \rangle$	$\langle 53, 43, 39, 54 \rangle$	$\langle 42, 10, 28, 37 \rangle$	$\langle 23, 50, 25, 26 \rangle$	$\langle 23, 10, 57, 05 \rangle$	$\langle 03, 52, 56, 16 \rangle$	$\langle 30, 10, 58, 47 \rangle$	$\langle 30, 52, 29, 27 \rangle$
$\langle 50, 52, 14, 15 \rangle$	$\langle 01, 42, 07, 45 \rangle$	$\langle 40, 31, 26, 55 \rangle$	$\langle 43, 12, 55, 59 \rangle$	$\langle 01, 23, 15, 55 \rangle$			

$n = 39$ :

$\langle \infty, 10, 39, 08 \rangle$	$\langle \infty, 11, 37, 36 \rangle$	$\langle \infty, 33, 04, 45 \rangle$	$\langle 02, 03, 04, a_1 \rangle$	$\langle 40, 11, 57, a_1 \rangle$	$\langle 42, 03, 56, a_2 \rangle$	$\langle 40, 01, 14, a_2 \rangle$	$\langle 10, 51, 34, a_3 \rangle$
$\langle 52, 33, 27, a_3 \rangle$	$\langle 53, 31, 48, 07 \rangle$	$\langle 51, 01, 24, 28 \rangle$	$\langle 43, 31, 49, 09 \rangle$	$\langle 00, 32, 05, 45 \rangle$	$\langle 01, 22, 25, 49 \rangle$	$\langle 43, 40, 56, 06 \rangle$	$\langle 11, 42, 08, 06 \rangle$
$\langle 41, 52, 26, 34 \rangle$	$\langle 10, 03, 45, 06 \rangle$	$\langle 10, 23, 19, 14 \rangle$	$\langle 51, 43, 07, 35 \rangle$	$\langle 32, 03, 08, 47 \rangle$	$\langle 03, 13, 28, 49 \rangle$	$\langle 42, 40, 34, 54 \rangle$	$\langle 33, 10, 07, 16 \rangle$
$\langle 01, 41, 09, 15 \rangle$	$\langle 50, 40, 26, 29 \rangle$	$\langle 53, 21, 26, 25 \rangle$	$\langle 42, 51, 26, 58 \rangle$	$\langle 42, 52, 46, 29 \rangle$	$\langle 00, 52, 15, 37 \rangle$	$\langle 53, 02, 09, 55 \rangle$	$\langle 11, 12, 17, 07 \rangle$
$\langle 50, 12, 09, 37 \rangle$	$\langle 10, 50, 17, 18 \rangle$	$\langle 30, 31, 25, 18 \rangle$	$\langle 22, 41, 44, 39 \rangle$	$\langle 50, 32, 15, 28 \rangle$	$\langle 30, 03, 14, 48 \rangle$	$\langle 02, 13, 34, 48 \rangle$	

$n = 41$ :

$\langle \infty, 30, 55, 14 \rangle$	$\langle \infty, 23, 59, 08 \rangle$	$\langle \infty, 11, 06, 37 \rangle$	$\langle 41, 50, 44, a_1 \rangle$	$\langle 52, 53, 25, a_1 \rangle$	$\langle 52, 03, 26, a_2 \rangle$	$\langle 41, 40, 54, a_2 \rangle$	$\langle 12, 33, 25, a_3 \rangle$
$\langle 31, 10, 17, a_3 \rangle$	$\langle 12, 43, 14, a_4 \rangle$	$\langle 21, 50, 35, a_4 \rangle$	$\langle 21, 40, 15, a_5 \rangle$	$\langle 42, 23, 04, a_5 \rangle$	$\langle 21, 31, 08, 55 \rangle$	$\langle 10, 21, 44, 59 \rangle$	$\langle 21, 13, 48, 27 \rangle$
$\langle 22, 21, 14, 09 \rangle$	$\langle 02, 12, 07, 06 \rangle$	$\langle 02, 51, 27, 58 \rangle$	$\langle 01, 03, 57, 59 \rangle$	$\langle 42, 33, 57, 49 \rangle$	$\langle 22, 00, 15, 58 \rangle$	$\langle 11, 31, 46, 15 \rangle$	$\langle 50, 53, 58, 26 \rangle$
$\langle 42, 22, 28, 05 \rangle$	$\langle 50, 42, 17, 27 \rangle$	$\langle 23, 41, 57, 09 \rangle$	$\langle 40, 23, 06, 26 \rangle$	$\langle 33, 53, 54, 55 \rangle$	$\langle 40, 42, 59, 45 \rangle$	$\langle 40, 50, 28, 37 \rangle$	$\langle 02, 50, 46, 14 \rangle$
$\langle 13, 01, 26, 19 \rangle$	$\langle 01, 32, 09, 46 \rangle$	$\langle 13, 40, 57, 39 \rangle$	$\langle 13, 00, 28, 55 \rangle$	$\langle 13, 41, 24, 38 \rangle$	$\langle 42, 51, 24, 08 \rangle$	$\langle 50, 13, 06, 08 \rangle$	$\langle 12, 30, 36, 39 \rangle$
$\langle 00, 32, 04, 29 \rangle$							

$n = 43$ :

$\langle \infty, 42, 34, 28 \rangle$	$\langle \infty, 11, 49, 36 \rangle$	$\langle \infty, 53, 05, 37 \rangle$	$\langle 22, 33, 46, a_1 \rangle$	$\langle 20, 31, 55, a_1 \rangle$	$\langle 20, 53, 44, a_2 \rangle$	$\langle 41, 12, 25, a_2 \rangle$	$\langle 03, 50, 57, a_3 \rangle$
$\langle 41, 32, 15, a_3 \rangle$	$\langle 53, 52, 07, a_4 \rangle$	$\langle 20, 41, 26, a_4 \rangle$	$\langle 30, 42, 44, a_5 \rangle$	$\langle 31, 33, 06, a_5 \rangle$	$\langle 40, 11, 06, a_6 \rangle$	$\langle 43, 22, 07, a_6 \rangle$	$\langle 03, 10, 44, a_7 \rangle$
$\langle 21, 22, 17, a_7 \rangle$	$\langle 13, 31, 44, 29 \rangle$	$\langle 31, 03, 08, 24 \rangle$	$\langle 52, 00, 04, 56 \rangle$	$\langle 32, 30, 09, 08 \rangle$	$\langle 21, 30, 38, 24 \rangle$	$\langle 31, 23, 25, 58 \rangle$	$\langle 01, 51, 17, 19 \rangle$
$\langle 53, 02, 25, 16 \rangle$	$\langle 10, 00, 46, 27 \rangle$	$\langle 32, 12, 54, 29 \rangle$	$\langle 02, 33, 37, 34 \rangle$	$\langle 20, 43, 36, 08 \rangle$	$\langle 22, 40, 55, 49 \rangle$	$\langle 32, 42, 16, 58 \rangle$	$\langle 22, 03, 29, 18 \rangle$
$\langle 01, 23, 15, 34 \rangle$	$\langle 22, 50, 47, 09 \rangle$	$\langle 01, 21, 08, 44 \rangle$	$\langle 20, 01, 49, 07 \rangle$	$\langle 00, 22, 25, 28 \rangle$	$\langle 53, 41, 27, 56 \rangle$	$\langle 13, 03, 49, 58 \rangle$	$\langle 31, 30, 07, 28 \rangle$
$\langle 40, 23, 29, 45 \rangle$	$\langle 00, 03, 45, 59 \rangle$	$\langle 01, 12, 05, 06 \rangle$					

$n = 45$ :

$\langle \infty, 03, 37, 26 \rangle$	$\langle \infty, 40, 54, 35 \rangle$	$\langle \infty, 02, 09, 38 \rangle$	$\langle 03, 02, 27, a_1 \rangle$	$\langle 50, 21, 16, a_1 \rangle$	$\langle 41, 30, 07, a_2 \rangle$	$\langle 32, 43, 46, a_2 \rangle$	$\langle 20, 11, 16, a_3 \rangle$
$\langle 33, 02, 54, a_3 \rangle$	$\langle 02, 53, 25, a_4 \rangle$	$\langle 01, 20, 44, a_4 \rangle$	$\langle 03, 42, 57, a_5 \rangle$	$\langle 40, 01, 25, a_5 \rangle$	$\langle 40, 23, 16, a_6 \rangle$	$\langle 52, 41, 54, a_6 \rangle$	$\langle 02, 30, 14, a_7 \rangle$
$\langle 31, 23, 15, a_7 \rangle$	$\langle 53, 11, 15, a_8 \rangle$	$\langle 00, 52, 57, a_8 \rangle$	$\langle 03, 31, 45, a_9 \rangle$	$\langle 32, 10, 04, a_9 \rangle$	$\langle 20, 43, 49, 55 \rangle$	$\langle 41, 53, 18, 04 \rangle$	$\langle 20, 53, 09, 37 \rangle$
$\langle 22, 01, 55, 48 \rangle$	$\langle 02, 43, 45, 47 \rangle$	$\langle 11, 12, 07, 59 \rangle$	$\langle 31, 21, 56, 39 \rangle$	$\langle 30, 50, 17, 29 \rangle$	$\langle 30, 33, 34, 08 \rangle$	$\langle 01, 00, 18, 34 \rangle$	$\langle 40, 52, 26, 48 \rangle$
$\langle 02, 52, 19, 46 \rangle$	$\langle 41, 32, 56, 38 \rangle$	$\langle 01, 42, 46, 39 \rangle$	$\langle 10, 03, 58, 17 \rangle$	$\langle 43, 41, 34, 09 \rangle$	$\langle 43, 23, 28, 56 \rangle$	$\langle 00, 10, 16, 15 \rangle$	$\langle 20, 33, 48, 29 \rangle$
$\langle 52, 32, 14, 45 \rangle$	$\langle 23, 01, 04, 59 \rangle$	$\langle 20, 02, 18, 39 \rangle$	$\langle 21, 41, 57, 48 \rangle$	$\langle 01, 32, 35, 07 \rangle$			

$n = 47$ :

$\langle \infty, 43, 55, 44 \rangle$	$\langle \infty, 31, 47, 46 \rangle$	$\langle \infty, 40, 39, 08 \rangle$	$\langle 40, 41, 27, a_1 \rangle$	$\langle 02, 33, 05, a_1 \rangle$	$\langle 23, 12, 46, a_2 \rangle$	$\langle 20, 31, 34, a_2 \rangle$	$\langle 23, 02, 57, a_3 \rangle$
$\langle 40, 31, 56, a_3 \rangle$	$\langle 41, 02, 24, a_4 \rangle$	$\langle 53, 20, 57, a_4 \rangle$	$\langle 20, 41, 55, a_5 \rangle$	$\langle 52, 43, 07, a_5 \rangle$	$\langle 42, 30, 24, a_6 \rangle$	$\langle 51, 33, 55, a_6 \rangle$	$\langle 02, 40, 16, a_7 \rangle$
$\langle 33, 01, 24, a_7 \rangle$	$\langle 20, 23, 04, a_8 \rangle$	$\langle 11, 02, 07, a_8 \rangle$	$\langle 50, 43, 45, a_9 \rangle$	$\langle 51, 32, 17, a_9 \rangle$	$\langle 43, 31, 04, a_{10} \rangle$	$\langle 42, 40, 25, a_{10} \rangle$	$\langle 40, 32, 07, a_{11} \rangle$
$\langle 13, 11, 24, a_{11} \rangle$	$\langle 51, 01, 58, 36 \rangle$	$\langle 00, 42, 09, 58 \rangle$	$\langle 51, 52, 44, 18 \rangle$	$\langle 02, 43, 58, 26 \rangle$	$\langle 40, 12, 59, 14 \rangle$	$\langle 41, 01, 19, 35 \rangle$	$\langle 13, 21, 26, 29 \rangle$
$\langle 12, 02, 06, 25 \rangle$	$\langle 10, 41, 25, 58 \rangle$	$\langle 11, 42, 59, 39 \rangle$	$\langle 30, 13, 09, 59 \rangle$	$\langle 51, 02, 46, 38 \rangle$	$\langle 40, 03, 29, 06 \rangle$	$\langle 23, 01, 59, 07 \rangle$	$\langle 10, 51, 15, 27 \rangle$
$\langle 22, 02, 55, 08 \rangle$	$\langle 12, 13, 44, 19 \rangle$	$\langle 30, 40, 26, 37 \rangle$	$\langle 43, 30, 36, 38 \rangle$	$\langle 10, 30, 34, 48 \rangle$	$\langle 53, 33, 47, 38 \rangle$	$\langle 03, 13, 55, 38 \rangle$	

$n = 49$ :

$\langle \infty, 52, 16, 57 \rangle$	$\langle \infty, 31, 410, 55 \rangle$	$\langle \infty, 00, 311, 18 \rangle$	$\langle \infty, 43, 29, 24 \rangle$	$\langle 50, 41, 55, a_1 \rangle$	$\langle 12, 23, 24, a_1 \rangle$	$\langle 31, 13, 211, 58 \rangle$	$\langle 20, 33, 510, 46 \rangle$
$\langle 40, 11, 59, 07 \rangle$	$\langle 00, 23, 14, 510 \rangle$	$\langle 12, 22, 19, 35 \rangle$	$\langle 10, 21, 26, 49 \rangle$	$\langle 03, 23, 110, 59 \rangle$	$\langle 21, 53, 07, 14 \rangle$	$\langle 43, 31, 26, 08 \rangle$	$\langle 32, 41, 49, 16 \rangle$
$\langle 01, 22, 14, 211 \rangle$	$\langle 11, 42, 17, 58 \rangle$	$\langle 30, 23, 16, 55 \rangle$	$\langle 31, 33, 48, 28 \rangle$	$\langle 01, 51, 27, 510 \rangle$	$\langle 40, 50, 34, 49 \rangle$	$\langle 13, 40, 36, 48 \rangle$	$\langle 01, 53, 59, 55 \rangle$
$\langle 30, 02, 28, 27 \rangle$	$\langle 30, 11, 510, 18 \rangle$	$\langle 52, 30, 45, 58 \rangle$	$\langle 50, 33, 57, 311 \rangle$	$\langle 00, 02, 17, 38 \rangle$	$\langle 20, 21, 311, 05 \rangle$	$\langle 12, 00, 410, 211 \rangle$	$\langle 02, 53, 24, 58 \rangle$
$\langle 13, 03, 57, 29 \rangle$	$\langle 22, 43, 111, 55 \rangle$	$\langle 12, 13, 55, 511 \rangle$	$\langle 00, 20, 55, 49 \rangle$	$\langle 00, 03, 511, 010 \rangle$	$\langle 01, 23, 411, 26 \rangle$	$\langle 22, 03, 25, 36 \rangle$	$\langle 22, 40, 411, 16 \rangle^s$
$\langle 00, 52, 47, 37 \rangle$	$\langle 10, 31, 34, 16 \rangle$	$\langle 01, 21, 39, 44 \rangle$	$\langle 51, 32, 210, 110 \rangle$	$\langle 31, 32, 04, 011 \rangle$	$\langle 02, 22, 210, 49 \rangle$	$\langle 02, 33, 44, 110 \rangle$	$\langle 00, 30, 04, 34 \rangle^s$
$\langle 01, 31, 05, 35 \rangle^s$	$\langle 02, 32, 06, 36 \rangle^s$	$\langle 03, 33, 07, 37 \rangle^s$					

Note that each of the codewords marked  $s$  only generates three codewords. ■

**Lemma 8.5:**  $T(2, 29; 2, n; 6) = 7n$  for each odd  $n$  and  $29 \leq n \leq 57$ .

*Proof:* Let  $X_1 = (\mathbb{Z}_7 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $29 \leq n \leq 41$ , let  $X_2 = (\mathbb{Z}_7 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-28\})$ ; for  $43 \leq n \leq 55$ , let  $X_2 = (\mathbb{Z}_7 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-42\})$ ; for  $n = 57$ , let  $X_2 = (\mathbb{Z}_7 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ . Denote  $X = X_1 \cup X_2$ . The desired codes of size  $7n$  are constructed on  $\mathbb{Z}_2^X$  and the base codewords are listed as follows.

$n = 29$ :

$\langle \infty, 20, 25, 54 \rangle$	$\langle \infty, 01, 07, 06 \rangle$	$\langle 62, 53, 35, a_1 \rangle$	$\langle 01, 60, 14, a_1 \rangle$	$\langle 22, 53, 27, 16 \rangle$	$\langle 22, 23, 25, 26 \rangle$	$\langle 13, 21, 05, 04 \rangle$	$\langle 10, 03, 67, 54 \rangle$
$\langle 00, 61, 16, 46 \rangle$	$\langle 12, 22, 14, 35 \rangle$	$\langle 32, 13, 67, 25 \rangle$	$\langle 10, 51, 27, 17 \rangle$	$\langle 02, 42, 24, 56 \rangle$	$\langle 01, 41, 67, 36 \rangle$	$\langle 01, 51, 05, 65 \rangle$	$\langle 10, 12, 57, 37 \rangle$
$\langle 21, 22, 66, 64 \rangle$	$\langle 10, 00, 25, 64 \rangle$	$\langle 30, 11, 27, 67 \rangle$	$\langle 22, 33, 55, 05 \rangle$	$\langle 10, 13, 45, 36 \rangle$	$\langle 13, 53, 26, 17 \rangle$	$\langle 12, 32, 27, 44 \rangle$	$\langle 11, 01, 34, 04 \rangle$
$\langle 20, 41, 56, 05 \rangle$	$\langle 00, 20, 06, 65 \rangle$	$\langle 22, 63, 56, 07 \rangle$	$\langle 13, 63, 24, 34 \rangle$	$\langle 00, 13, 14, 66 \rangle$			

$n = 31$ :

$\langle \infty, 50, 45, 54 \rangle$	$\langle \infty, 31, 07, 26 \rangle$	$\langle 13, 12, 17, a_1 \rangle$	$\langle 41, 20, 34, a_1 \rangle$	$\langle 20, 31, 54, a_2 \rangle$	$\langle 53, 42, 66, a_2 \rangle$	$\langle 63, 42, 37, a_3 \rangle$	$\langle 61, 20, 44, a_3 \rangle$
$\langle 50, 11, 67, 27 \rangle$	$\langle 60, 41, 67, 26 \rangle$	$\langle 52, 23, 05, 07 \rangle$	$\langle 22, 20, 07, 57 \rangle$	$\langle 42, 23, 57, 04 \rangle$	$\langle 40, 10, 16, 26 \rangle$	$\langle 41, 50, 65, 25 \rangle$	$\langle 61, 33, 16, 35 \rangle$
$\langle 60, 43, 54, 16 \rangle$	$\langle 12, 32, 24, 34 \rangle$	$\langle 31, 62, 37, 34 \rangle$	$\langle 42, 03, 24, 45 \rangle$	$\langle 23, 13, 14, 54 \rangle$	$\langle 22, 62, 35, 56 \rangle$	$\langle 60, 03, 65, 17 \rangle$	$\langle 33, 03, 27, 66 \rangle$
$\langle 31, 51, 65, 66 \rangle$	$\langle 33, 53, 65, 56 \rangle$	$\langle 30, 40, 14, 65 \rangle$	$\langle 50, 02, 35, 46 \rangle$	$\langle 12, 21, 26, 66 \rangle$	$\langle 21, 51, 64, 17 \rangle$	$\langle 01, 12, 05, 65 \rangle$	

$n = 33$ :

$\langle \infty, 30, 34, 35 \rangle$	$\langle \infty, 11, 66, 47 \rangle$	$\langle 52, 43, 45, a_1 \rangle$	$\langle 30, 31, 44, a_1 \rangle$	$\langle 62, 03, 07, a_2 \rangle$	$\langle 60, 01, 24, a_2 \rangle$	$\langle 23, 02, 56, a_3 \rangle$	$\langle 50, 31, 37, a_3 \rangle$
$\langle 42, 03, 37, a_4 \rangle$	$\langle 40, 31, 64, a_4 \rangle$	$\langle 50, 11, 57, a_5 \rangle$	$\langle 42, 13, 44, a_5 \rangle$	$\langle 11, 23, 67, 05 \rangle$	$\langle 11, 01, 25, 54 \rangle$	$\langle 21, 01, 55, 16 \rangle$	$\langle 03, 63, 14, 44 \rangle$
$\langle 10, 13, 05, 55 \rangle$	$\langle 11, 32, 55, 46 \rangle$	$\langle 10, 03, 27, 25 \rangle$	$\langle 40, 12, 25, 06 \rangle$	$\langle 30, 63, 37, 17 \rangle$	$\langle 21, 22, 14, 46 \rangle$	$\langle 30, 62, 27, 67 \rangle$	$\langle 32, 20, 14, 06 \rangle$
$\langle 10, 20, 64, 45 \rangle$	$\langle 02, 42, 06, 05 \rangle$	$\langle 20, 22, 47, 67 \rangle$	$\langle 12, 22, 54, 34 \rangle$	$\langle 22, 61, 65, 07 \rangle$	$\langle 00, 30, 46, 26 \rangle$	$\langle 03, 43, 57, 66 \rangle$	$\langle 13, 63, 25, 66 \rangle$
$\langle 01, 03, 04, 67 \rangle$							

$n = 35$ :

$\langle \infty, 12, 05, 14 \rangle$	$\langle \infty, 11, 16, 17 \rangle$	$\langle 63, 22, 67, a_1 \rangle$	$\langle 60, 61, 04, a_1 \rangle$	$\langle 52, 43, 65, a_2 \rangle$	$\langle 50, 61, 14, a_2 \rangle$	$\langle 51, 13, 54, a_3 \rangle$	$\langle 00, 02, 26, a_3 \rangle$
$\langle 33, 32, 24, a_4 \rangle$	$\langle 60, 21, 45, a_4 \rangle$	$\langle 52, 13, 47, a_5 \rangle$	$\langle 30, 51, 45, a_5 \rangle$	$\langle 50, 31, 24, a_6 \rangle$	$\langle 53, 42, 56, a_6 \rangle$	$\langle 10, 01, 37, a_7 \rangle$	$\langle 62, 43, 66, a_7 \rangle$
$\langle 12, 33, 66, 45 \rangle$	$\langle 21, 51, 37, 35 \rangle$	$\langle 33, 41, 05, 06 \rangle$	$\langle 03, 20, 17, 65 \rangle$	$\langle 30, 63, 37, 17 \rangle$	$\langle 32, 52, 17, 64 \rangle$	$\langle 01, 03, 67, 34 \rangle$	$\langle 53, 11, 66, 37 \rangle$
$\langle 43, 50, 45, 26 \rangle$	$\langle 41, 31, 14, 56 \rangle$	$\langle 02, 50, 05, 36 \rangle$	$\langle 22, 60, 64, 65 \rangle$	$\langle 23, 50, 16, 34 \rangle$	$\langle 41, 62, 17, 45 \rangle$	$\langle 23, 43, 05, 44 \rangle$	$\langle 20, 30, 36, 67 \rangle$
$\langle 51, 52, 26, 46 \rangle$	$\langle 32, 20, 55, 37 \rangle$	$\langle 00, 42, 24, 64 \rangle$					

$n = 37$ :

$\langle \infty, 21, 27, 26 \rangle$	$\langle \infty, 10, 15, 14 \rangle$	$\langle 52, 53, 25, a_1 \rangle$	$\langle 50, 51, 64, a_1 \rangle$	$\langle 60, 01, 24, a_2 \rangle$	$\langle 42, 53, 55, a_2 \rangle$	$\langle 11, 60, 14, a_3 \rangle$	$\langle 52, 03, 15, a_3 \rangle$
$\langle 01, 40, 34, a_4 \rangle$	$\langle 42, 03, 35, a_4 \rangle$	$\langle 22, 63, 46, a_5 \rangle$	$\langle 20, 61, 35, a_5 \rangle$	$\langle 62, 43, 27, a_6 \rangle$	$\langle 21, 40, 14, a_6 \rangle$	$\langle 40, 31, 65, a_7 \rangle$	$\langle 52, 43, 24, a_7 \rangle$
$\langle 20, 42, 04, a_8 \rangle$	$\langle 21, 23, 45, a_8 \rangle$	$\langle 11, 43, 47, a_9 \rangle$	$\langle 30, 32, 36, a_9 \rangle$	$\langle 01, 42, 26, 17 \rangle$	$\langle 10, 00, 45, 26 \rangle$	$\langle 20, 32, 47, 15 \rangle$	$\langle 02, 62, 04, 54 \rangle$
$\langle 21, 42, 06, 64 \rangle$	$\langle 13, 03, 65, 04 \rangle$	$\langle 10, 50, 57, 67 \rangle$	$\langle 02, 32, 46, 27 \rangle$	$\langle 31, 43, 07, 14 \rangle$	$\langle 11, 33, 56, 26 \rangle$	$\langle 01, 21, 05, 15 \rangle$	$\langle 01, 41, 67, 36 \rangle$
$\langle 03, 23, 36, 34 \rangle$	$\langle 00, 20, 66, 56 \rangle$	$\langle 03, 43, 46, 17 \rangle$	$\langle 20, 63, 17, 57 \rangle$	$\langle 02, 32, 25, 07 \rangle$			



$n = 39$ :

$\langle \infty, 61, 67, 66 \rangle$	$\langle \infty, 10, 15, 24 \rangle$	$\langle 41, 40, 55, a_1 \rangle$	$\langle 42, 43, 16, a_1 \rangle$	$\langle 50, 61, 14, a_2 \rangle$	$\langle 32, 43, 05, a_2 \rangle$	$\langle 52, 43, 66, a_3 \rangle$	$\langle 10, 01, 35, a_3 \rangle$
$\langle 22, 53, 07, a_4 \rangle$	$\langle 41, 10, 04, a_4 \rangle$	$\langle 23, 52, 54, a_5 \rangle$	$\langle 41, 00, 56, a_5 \rangle$	$\langle 23, 42, 45, a_6 \rangle$	$\langle 10, 61, 54, a_6 \rangle$	$\langle 41, 42, 54, a_7 \rangle$	$\langle 00, 23, 35, a_7 \rangle$
$\langle 23, 21, 64, a_8 \rangle$	$\langle 60, 32, 16, a_8 \rangle$	$\langle 60, 02, 66, a_9 \rangle$	$\langle 61, 43, 44, a_9 \rangle$	$\langle 02, 20, 15, a_{10} \rangle$	$\langle 61, 03, 57, a_{10} \rangle$	$\langle 43, 11, 35, a_{11} \rangle$	$\langle 22, 20, 04, a_{11} \rangle$
$\langle 40, 20, 36, 27 \rangle$	$\langle 20, 23, 57, 05 \rangle$	$\langle 23, 43, 37, 26 \rangle$	$\langle 23, 63, 65, 67 \rangle$	$\langle 23, 10, 14, 56 \rangle$	$\langle 30, 00, 47, 27 \rangle$	$\langle 11, 01, 55, 05 \rangle$	$\langle 21, 01, 46, 56 \rangle$
$\langle 22, 11, 14, 37 \rangle$	$\langle 11, 62, 57, 67 \rangle$	$\langle 41, 32, 07, 36 \rangle$	$\langle 12, 42, 65, 05 \rangle$	$\langle 00, 43, 24, 36 \rangle$	$\langle 12, 22, 44, 47 \rangle$	$\langle 02, 23, 44, 36 \rangle$	

$n = 41$ :

$\langle \infty, 21, 15, 14 \rangle$	$\langle \infty, 42, 17, 46 \rangle$	$\langle 21, 20, 64, a_1 \rangle$	$\langle 22, 23, 56, a_1 \rangle$	$\langle 12, 23, 66, a_2 \rangle$	$\langle 21, 10, 44, a_2 \rangle$	$\langle 03, 52, 45, a_3 \rangle$	$\langle 20, 41, 44, a_3 \rangle$
$\langle 03, 12, 14, a_4 \rangle$	$\langle 10, 41, 16, a_4 \rangle$	$\langle 42, 03, 07, a_5 \rangle$	$\langle 11, 40, 45, a_5 \rangle$	$\langle 33, 62, 15, a_6 \rangle$	$\langle 01, 20, 34, a_6 \rangle$	$\langle 60, 51, 64, a_7 \rangle$	$\langle 13, 32, 37, a_7 \rangle$
$\langle 01, 52, 55, a_8 \rangle$	$\langle 13, 30, 26, a_8 \rangle$	$\langle 43, 21, 04, a_9 \rangle$	$\langle 12, 40, 25, a_9 \rangle$	$\langle 23, 31, 46, a_{10} \rangle$	$\langle 22, 20, 04, a_{10} \rangle$	$\langle 03, 21, 17, a_{11} \rangle$	$\langle 50, 42, 16, a_{11} \rangle$
$\langle 53, 51, 05, a_{12} \rangle$	$\langle 42, 20, 36, a_{12} \rangle$	$\langle 62, 10, 04, a_{13} \rangle$	$\langle 33, 21, 07, a_{13} \rangle$	$\langle 30, 60, 57, 37 \rangle$	$\langle 30, 50, 25, 17 \rangle$	$\langle 23, 43, 14, 35 \rangle$	$\langle 61, 31, 35, 66 \rangle$
$\langle 61, 33, 67, 16 \rangle$	$\langle 63, 33, 14, 65 \rangle$	$\langle 00, 10, 56, 35 \rangle$	$\langle 32, 52, 24, 47 \rangle$	$\langle 33, 23, 17, 26 \rangle$	$\langle 21, 01, 47, 37 \rangle$	$\langle 52, 20, 35, 37 \rangle$	$\langle 32, 61, 56, 46 \rangle$
$\langle 02, 12, 34, 45 \rangle$							

$n = 43$ :

$\langle \infty, 23, 58, 59 \rangle$	$\langle \infty, 02, 36, 24 \rangle$	$\langle \infty, 40, 25, 57 \rangle$	$\langle 53, 42, 45, a_1 \rangle$	$\langle 50, 51, 46, a_1 \rangle$	$\langle 31, 51, 06, 55 \rangle$	$\langle 20, 60, 49, 59 \rangle$	$\langle 00, 43, 26, 48 \rangle$
$\langle 42, 40, 56, 46 \rangle$	$\langle 50, 32, 34, 47 \rangle$	$\langle 22, 51, 28, 47 \rangle$	$\langle 61, 63, 49, 37 \rangle$	$\langle 23, 00, 34, 27 \rangle$	$\langle 42, 11, 59, 24 \rangle$	$\langle 13, 53, 59, 26 \rangle$	$\langle 03, 40, 34, 44 \rangle$
$\langle 32, 52, 45, 18 \rangle$	$\langle 33, 02, 47, 49 \rangle$	$\langle 23, 61, 26, 46 \rangle$	$\langle 31, 01, 17, 69 \rangle$	$\langle 32, 42, 69, 16 \rangle$	$\langle 00, 11, 47, 18 \rangle$	$\langle 50, 42, 37, 58 \rangle$	$\langle 61, 03, 66, 05 \rangle$
$\langle 60, 02, 34, 09 \rangle$	$\langle 01, 43, 27, 28 \rangle$	$\langle 60, 53, 65, 04 \rangle$	$\langle 20, 01, 29, 65 \rangle$	$\langle 30, 51, 25, 54 \rangle$	$\langle 02, 43, 25, 44 \rangle$	$\langle 11, 21, 04, 54 \rangle$	$\langle 22, 23, 48, 14 \rangle$
$\langle 20, 00, 58, 35 \rangle$	$\langle 31, 42, 39, 54 \rangle$	$\langle 00, 03, 36, 68 \rangle$	$\langle 22, 21, 05, 55 \rangle$	$\langle 50, 21, 08, 36 \rangle$	$\langle 33, 20, 27, 57 \rangle$	$\langle 61, 53, 28, 09 \rangle$	$\langle 13, 30, 09, 55 \rangle$
$\langle 62, 22, 16, 27 \rangle$	$\langle 62, 41, 38, 58 \rangle$	$\langle 02, 23, 45, 57 \rangle$					

$n = 45$ :

$\langle \infty, 22, 29, 28 \rangle$	$\langle \infty, 20, 24, 25 \rangle$	$\langle \infty, 21, 46, 37 \rangle$	$\langle 50, 51, 64, a_1 \rangle$	$\langle 12, 13, 15, a_1 \rangle$	$\langle 31, 20, 54, a_2 \rangle$	$\langle 42, 53, 26, a_2 \rangle$	$\langle 40, 61, 64, a_3 \rangle$
$\langle 32, 53, 45, a_3 \rangle$	$\langle 00, 50, 29, 35 \rangle$	$\langle 01, 41, 59, 25 \rangle$	$\langle 21, 52, 29, 25 \rangle$	$\langle 31, 43, 07, 09 \rangle$	$\langle 10, 22, 26, 58 \rangle$	$\langle 12, 43, 54, 67 \rangle$	$\langle 01, 33, 45, 68 \rangle$
$\langle 10, 41, 36, 04 \rangle$	$\langle 21, 63, 66, 68 \rangle$	$\langle 10, 42, 35, 37 \rangle$	$\langle 03, 53, 67, 16 \rangle$	$\langle 10, 03, 29, 57 \rangle$	$\langle 30, 52, 27, 38 \rangle$	$\langle 20, 01, 56, 29 \rangle$	$\langle 13, 43, 28, 36 \rangle$
$\langle 21, 23, 27, 19 \rangle$	$\langle 30, 32, 19, 67 \rangle$	$\langle 01, 52, 57, 16 \rangle$	$\langle 21, 11, 64, 48 \rangle$	$\langle 12, 62, 45, 08 \rangle$	$\langle 31, 02, 48, 27 \rangle$	$\langle 10, 01, 27, 06 \rangle$	$\langle 02, 62, 19, 14 \rangle$
$\langle 00, 13, 69, 45 \rangle$	$\langle 03, 63, 24, 54 \rangle$	$\langle 02, 42, 16, 66 \rangle$	$\langle 20, 23, 48, 06 \rangle$	$\langle 11, 12, 49, 04 \rangle$	$\langle 30, 01, 36, 37 \rangle$	$\langle 12, 63, 38, 64 \rangle$	$\langle 00, 60, 44, 58 \rangle$
$\langle 42, 33, 39, 04 \rangle$	$\langle 01, 51, 65, 58 \rangle$	$\langle 32, 03, 55, 47 \rangle$	$\langle 20, 43, 38, 59 \rangle$	$\langle 00, 43, 15, 65 \rangle$			

$n = 47$ :

$\langle \infty, 32, 38, 69 \rangle$	$\langle \infty, 31, 57, 16 \rangle$	$\langle \infty, 30, 35, 04 \rangle$	$\langle 61, 20, 65, a_1 \rangle$	$\langle 12, 43, 37, a_1 \rangle$	$\langle 53, 02, 25, a_2 \rangle$	$\langle 51, 40, 04, a_2 \rangle$	$\langle 23, 02, 15, a_3 \rangle$
$\langle 51, 30, 67, a_3 \rangle$	$\langle 22, 33, 54, a_4 \rangle$	$\langle 61, 30, 06, a_4 \rangle$	$\langle 01, 20, 35, a_5 \rangle$	$\langle 03, 02, 16, a_5 \rangle$	$\langle 21, 01, 28, 65 \rangle$	$\langle 12, 00, 67, 09 \rangle$	$\langle 41, 43, 27, 19 \rangle$
$\langle 60, 20, 06, 19 \rangle$	$\langle 41, 22, 24, 34 \rangle$	$\langle 40, 33, 57, 08 \rangle$	$\langle 02, 12, 17, 19 \rangle$	$\langle 33, 13, 69, 47 \rangle$	$\langle 12, 30, 07, 48 \rangle$	$\langle 40, 50, 34, 48 \rangle$	$\langle 31, 32, 04, 67 \rangle$
$\langle 42, 13, 66, 25 \rangle$	$\langle 61, 31, 19, 48 \rangle$	$\langle 03, 43, 66, 68 \rangle$	$\langle 22, 50, 04, 69 \rangle$	$\langle 12, 20, 15, 08 \rangle$	$\langle 41, 52, 39, 16 \rangle$	$\langle 23, 40, 66, 27 \rangle$	$\langle 11, 53, 49, 58 \rangle$
$\langle 13, 21, 54, 45 \rangle$	$\langle 43, 61, 45, 69 \rangle$	$\langle 51, 13, 57, 48 \rangle$	$\langle 11, 20, 16, 48 \rangle$	$\langle 13, 40, 19, 44 \rangle$	$\langle 62, 22, 08, 55 \rangle$	$\langle 20, 42, 68, 26 \rangle$	$\langle 43, 52, 44, 28 \rangle$
$\langle 11, 33, 36, 14 \rangle$	$\langle 33, 10, 24, 49 \rangle$	$\langle 51, 22, 16, 65 \rangle$	$\langle 32, 41, 59, 54 \rangle$	$\langle 13, 10, 35, 46 \rangle$	$\langle 22, 01, 66, 67 \rangle$	$\langle 00, 20, 55, 27 \rangle$	

$n = 49$ :

$\langle \infty, 11, 04, 17 \rangle$	$\langle \infty, 30, 25, 58 \rangle$	$\langle \infty, 33, 26, 49 \rangle$	$\langle 00, 42, 04, a_1 \rangle$	$\langle 41, 53, 55, a_1 \rangle$	$\langle 13, 42, 07, a_2 \rangle$	$\langle 30, 51, 46, a_2 \rangle$	$\langle 42, 60, 17, a_3 \rangle$
$\langle 31, 03, 54, a_3 \rangle$	$\langle 41, 30, 35, a_4 \rangle$	$\langle 03, 02, 46, a_4 \rangle$	$\langle 40, 23, 54, a_5 \rangle$	$\langle 11, 62, 67, a_5 \rangle$	$\langle 53, 50, 57, a_6 \rangle$	$\langle 01, 12, 35, a_6 \rangle$	$\langle 13, 62, 34, a_7 \rangle$
$\langle 01, 20, 55, a_7 \rangle$	$\langle 52, 33, 55, 59 \rangle$	$\langle 50, 42, 27, 29 \rangle$	$\langle 60, 02, 56, 28 \rangle$	$\langle 50, 63, 16, 47 \rangle$	$\langle 12, 43, 06, 46 \rangle$	$\langle 21, 23, 48, 59 \rangle$	$\langle 61, 21, 17, 68 \rangle$
$\langle 01, 32, 47, 05 \rangle$	$\langle 30, 62, 38, 16 \rangle$	$\langle 03, 10, 65, 68 \rangle$	$\langle 51, 33, 05, 67 \rangle$	$\langle 11, 21, 64, 09 \rangle$	$\langle 22, 33, 39, 04 \rangle$	$\langle 40, 03, 15, 64 \rangle$	$\langle 52, 50, 19, 65 \rangle$
$\langle 22, 52, 49, 15 \rangle$	$\langle 10, 60, 09, 08 \rangle$	$\langle 22, 02, 24, 58 \rangle$	$\langle 13, 33, 57, 68 \rangle$	$\langle 51, 10, 36, 64 \rangle$	$\langle 21, 53, 29, 36 \rangle$	$\langle 11, 02, 14, 55 \rangle$	$\langle 51, 43, 56, 18 \rangle$
$\langle 40, 00, 34, 57 \rangle$	$\langle 23, 32, 24, 38 \rangle$	$\langle 31, 32, 09, 28 \rangle$	$\langle 62, 41, 66, 06 \rangle$	$\langle 23, 01, 34, 19 \rangle$	$\langle 51, 22, 38, 17 \rangle$	$\langle 50, 51, 26, 09 \rangle$	$\langle 63, 40, 49, 07 \rangle$
$\langle 00, 43, 25, 48 \rangle$							

$n = 51$ :

$\langle \infty, 52, 65, 08 \rangle$	$\langle \infty, 20, 66, 64 \rangle$	$\langle \infty, 23, 47, 19 \rangle$	$\langle 50, 51, 47, a_1 \rangle$	$\langle 23, 02, 45, a_1 \rangle$	$\langle 20, 53, 24, a_2 \rangle$	$\langle 12, 61, 66, a_2 \rangle$	$\langle 20, 32, 07, a_3 \rangle$
$\langle 41, 43, 06, a_3 \rangle$	$\langle 31, 60, 44, a_4 \rangle$	$\langle 32, 63, 67, a_4 \rangle$	$\langle 02, 61, 16, a_5 \rangle$	$\langle 20, 23, 44, a_5 \rangle$	$\langle 50, 42, 06, a_6 \rangle$	$\langle 31, 63, 45, a_6 \rangle$	$\langle 23, 00, 65, a_7 \rangle$
$\langle 01, 42, 34, a_7 \rangle$	$\langle 51, 00, 07, a_8 \rangle$	$\langle 63, 22, 05, a_8 \rangle$	$\langle 23, 11, 57, a_9 \rangle$	$\langle 40, 12, 56, a_9 \rangle$	$\langle 20, 41, 09, a_7 \rangle$	$\langle 22, 23, 17, 39 \rangle$	$\langle 13, 23, 67, 49 \rangle$
$\langle 12, 11, 69, 45 \rangle$	$\langle 42, 32, 28, 35 \rangle$	$\langle 00, 13, 19, 36 \rangle$	$\langle 32, 02, 54, 17 \rangle$	$\langle 33, 61, 36, 35 \rangle$	$\langle 02, 00, 66, 38 \rangle$	$\langle 00, 40, 68, 25 \rangle$	$\langle 32, 10, 44, 04 \rangle$
$\langle 12, 63, 39, 44 \rangle$	$\langle 53, 03, 58, 18 \rangle$	$\langle 62, 12, 59, 16 \rangle$	$\langle 10, 30, 16, 39 \rangle$	$\langle 13, 61, 45, 56 \rangle$	$\langle 10, 20, 25, 59 \rangle$	$\langle 01, 63, 49, 04 \rangle$	$\langle 42, 53, 48, 67 \rangle$
$\langle 10, 01, 24, 09 \rangle$	$\langle 03, 33, 34, 16 \rangle$	$\langle 50, 61, 38, 15 \rangle$	$\langle 30, 13, 05, 38 \rangle$	$\langle 11, 41, 26, 48 \rangle$	$\langle 33, 51, 24, 08 \rangle$	$\langle 40, 01, 58, 17 \rangle$	$\langle 10, 42, 47, 58 \rangle$
$\langle 31, 11, 28, 67 \rangle$	$\langle 51, 32, 55, 69 \rangle$	$\langle 01, 62, 64, 69 \rangle$					

$n = 53$ :

$\langle \infty, 11, 17, 46 \rangle$	$\langle \infty, 03, 19, 08 \rangle$	$\langle \infty, 00, 04, 35 \rangle$	$\langle 32, 33, 35, a_1 \rangle$	$\langle 21, 40, 06, a_1 \rangle$	$\langle 62, 03, 35, a_2 \rangle$	$\langle 51, 40, 04, a_2 \rangle$	$\langle 62, 13, 26, a_3 \rangle$
$\langle 41, 50, 55, a_3 \rangle$	$\langle 41, 10, 54, a_4 \rangle$	$\langle 63, 12, 16, a_4 \rangle$	$\langle 13, 52, 54, a_5 \rangle$	$\langle 00, 41, 16, a_5 \rangle$	$\langle 30, 22, 07, a_6 \rangle$	$\langle 21, 03, 15, a_6 \rangle$	$\langle 50, 51, 05, a_7 \rangle$
$\langle 03, 12, 27, a_7 \rangle$	$\langle 12, 10, 64, a_8 \rangle$	$\langle 43, 11, 27, a_8 \rangle$	$\langle 53, 61, 16, a_9 \rangle$	$\langle 30, 12, 47, a_9 \rangle$	$\langle 22, 10, 67, a_{10} \rangle$	$\langle 53, 31, 34, a_{10} \rangle$	$\langle 10, 42, 36, a_{11} \rangle$
$\langle 61, 63, 57, a_{11} \rangle$	$\langle 00, 63, 39, 08 \rangle$	$\langle 23, 51, 09, 34 \rangle$	$\langle 00, 22, 64, 06 \rangle$	$\langle 20, 60, 17, 09 \rangle$	$\langle 11, 62, 19, 54 \rangle$	$\langle 32, 11, 15, 57 \rangle$	$\langle 10, 13, 08, 66 \rangle$
$\langle 63, 33, 37, 36 \rangle$	$\langle 01, 11, 06, 49 \rangle$	$\langle 42, 11, 28, 38 \rangle$	$\langle 23, 03, 65, 29 \rangle$	$\langle 23, 33, 24, 68 \rangle$	$\langle 32, 60, 27, 45 \rangle$	$\langle 03, 20, 66, 69 \rangle$	$\langle 12, 52, 19, 26 \rangle$
$\langle 42, 62, 59, 04 \rangle$	$\langle 51, 22, 38, 44 \rangle$	$\langle 02, 12, 35, 48 \rangle$	$\langle 43, 02, 28, 07 \rangle$	$\langle 03, 40, 39, 55 \rangle$	$\langle 10, 60, 19, 48 \rangle$	$\langle 42, 51, 66, 35 \rangle$	$\langle 00, 10, 24, 28 \rangle$
$\langle 11, 60, 55, 67 \rangle$	$\langle 21, 01, 19, 68 \rangle$	$\langle 11, 12, 18, 69 \rangle$	$\langle 63, 40, 15, 18 \rangle$	$\langle 01, 13, 34, 27 \rangle$			

$n = 55$ :

$(\infty, 13, 09, 26)$	$(\infty, 02, 15, 47)$	$(\infty, 30, 38, 04)$	$(41, 63, 27, a_1)$	$(30, 02, 66, a_1)$	$(10, 22, 14, a_2)$	$(41, 53, 25, a_2)$	$(51, 12, 34, a_3)$
$(43, 20, 36, a_3)$	$(42, 03, 56, a_4)$	$(00, 21, 37, a_4)$	$(13, 12, 14, a_5)$	$(00, 31, 45, a_5)$	$(23, 10, 45, a_6)$	$(12, 21, 56, a_6)$	$(31, 32, 57, a_7)$
$(00, 33, 64, a_7)$	$(33, 01, 07, a_8)$	$(30, 22, 25, a_8)$	$(23, 42, 27, a_9)$	$(31, 50, 64, a_9)$	$(21, 02, 25, a_{10})$	$(10, 13, 37, a_{10})$	$(62, 40, 26, a_{11})$
$(13, 11, 04, a_{11})$	$(21, 62, 66, a_{12})$	$(50, 43, 35, a_{12})$	$(13, 40, 36, a_{13})$	$(01, 22, 37, a_{13})$	$(20, 11, 38, 07)$	$(10, 63, 07, 48)$	$(42, 13, 07, 24)$
$(02, 61, 55, 59)$	$(52, 12, 28, 08)$	$(62, 53, 29, 55)$	$(22, 43, 27, 69)$	$(23, 63, 38, 09)$	$(21, 03, 06, 38)$	$(13, 21, 46, 18)$	$(61, 50, 48, 56)$
$(10, 12, 29, 36)$	$(30, 60, 37, 18)$	$(00, 60, 59, 39)$	$(11, 31, 34, 69)$	$(12, 02, 09, 45)$	$(01, 61, 06, 09)$	$(11, 10, 39, 58)$	$(00, 20, 25, 54)$
$(53, 42, 08, 26)$	$(21, 61, 34, 28)$	$(11, 53, 35, 59)$	$(30, 12, 54, 39)$	$(21, 50, 65, 67)$	$(02, 22, 34, 08)$	$(03, 23, 44, 35)$	



$n = 57$ :

$\langle \infty, 20, 611, 25 \rangle$	$\langle \infty, 02, 27, 14 \rangle$	$\langle \infty, 51, 210, 09 \rangle$	$\langle \infty, 43, 56, 28 \rangle$	$\langle 13, 62, 34, a_1 \rangle$	$\langle 01, 20, 67, a_1 \rangle$	$\langle 02, 42, 66, 311 \rangle$	$\langle 01, 62, 15, 59 \rangle$
$\langle 30, 40, 210, 59 \rangle$	$\langle 01, 33, 35, 69 \rangle$	$\langle 12, 13, 29, 17 \rangle$	$\langle 01, 22, 05, 25 \rangle$	$\langle 22, 42, 57, 010 \rangle$	$\langle 23, 12, 210, 49 \rangle$	$\langle 03, 13, 65, 35 \rangle$	$\langle 13, 42, 38, 19 \rangle$
$\langle 10, 01, 44, 210 \rangle$	$\langle 40, 61, 14, 39 \rangle$	$\langle 42, 50, 18, 35 \rangle$	$\langle 60, 10, 69, 46 \rangle$	$\langle 10, 33, 38, 06 \rangle$	$\langle 60, 23, 04, 110 \rangle$	$\langle 12, 63, 010, 511 \rangle$	$\langle 03, 31, 011, 44 \rangle$
$\langle 11, 22, 17, 49 \rangle$	$\langle 20, 31, 08, 15 \rangle$	$\langle 53, 01, 68, 37 \rangle$	$\langle 42, 03, 56, 49 \rangle$	$\langle 42, 32, 05, 511 \rangle$	$\langle 21, 23, 67, 65 \rangle$	$\langle 31, 02, 34, 511 \rangle$	$\langle 40, 01, 65, 16 \rangle$
$\langle 60, 63, 310, 58 \rangle$	$\langle 33, 20, 510, 511 \rangle$	$\langle 23, 53, 46, 17 \rangle$	$\langle 01, 61, 610, 17 \rangle$	$\langle 30, 23, 311, 34 \rangle$	$\langle 12, 31, 38, 64 \rangle$	$\langle 41, 61, 36, 210 \rangle$	$\langle 30, 02, 08, 56 \rangle$
$\langle 22, 20, 210, 24 \rangle$	$\langle 01, 23, 26, 56 \rangle$	$\langle 00, 22, 08, 35 \rangle$	$\langle 20, 32, 36, 311 \rangle$	$\langle 30, 00, 45, 24 \rangle$	$\langle 10, 53, 07, 311 \rangle$	$\langle 21, 13, 58, 04 \rangle$	$\langle 30, 62, 47, 14 \rangle$
$\langle 62, 53, 110, 310 \rangle$	$\langle 31, 32, 48, 24 \rangle$	$\langle 30, 23, 37, 09 \rangle$	$\langle 01, 31, 611, 58 \rangle$	$\langle 01, 32, 06, 19 \rangle$	$\langle 51, 63, 59, 311 \rangle$	$\langle 40, 11, 46, 211 \rangle$	$\langle 00, 01, 57, 011 \rangle$
$\langle 00, 52, 27, 18 \rangle$							

■

**Lemma 8.6:**  $T(2, 33; 2, n; 6) = 8n$  for each odd  $n$  and  $33 \leq n \leq 65$ .

*Proof:* Let  $X_1 = (\mathbb{Z}_8 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $33 \leq n \leq 47$ , let  $X_2 = (\mathbb{Z}_8 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-32\})$ ; for  $49 \leq n \leq 63$ , let  $X_2 = (\mathbb{Z}_8 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-48\})$ ; for  $n = 65$ , let  $X_2 = (\mathbb{Z}_8 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ . Denote  $X = X_1 \cup X_2$ . The desired codes of size  $8n$  are constructed on  $\mathbb{Z}_2^X$  and the base codewords are listed as follows.

$n = 33$ :

$\langle \infty, 71, 26, 47 \rangle$	$\langle \infty, 00, 55, 04 \rangle$	$\langle 10, 61, 35, a_1 \rangle$	$\langle 42, 63, 34, a_1 \rangle$	$\langle 33, 10, 24, 17 \rangle$	$\langle 42, 00, 74, 05 \rangle$	$\langle 33, 63, 46, 37 \rangle$	$\langle 71, 62, 66, 56 \rangle$
$\langle 12, 70, 47, 24 \rangle$	$\langle 00, 60, 46, 47 \rangle$	$\langle 13, 51, 45, 57 \rangle$	$\langle 42, 52, 67, 25 \rangle$	$\langle 53, 33, 25, 64 \rangle$	$\langle 31, 21, 35, 44 \rangle$	$\langle 70, 22, 66, 17 \rangle$	$\langle 71, 11, 57, 74 \rangle$
$\langle 33, 02, 75, 26 \rangle$	$\langle 20, 31, 46, 57 \rangle$	$\langle 42, 23, 26, 45 \rangle$	$\langle 61, 11, 66, 45 \rangle$	$\langle 62, 61, 24, 05 \rangle$	$\langle 40, 70, 35, 55 \rangle$	$\langle 43, 21, 66, 37 \rangle$	$\langle 22, 71, 67, 35 \rangle$
$\langle 70, 61, 54, 06 \rangle$	$\langle 70, 23, 76, 25 \rangle$	$\langle 30, 32, 04, 06 \rangle$	$\langle 03, 32, 37, 65 \rangle$	$\langle 33, 72, 47, 74 \rangle$	$\langle 03, 10, 27, 46 \rangle$	$\langle 72, 73, 26, 14 \rangle$	$\langle 00, 43, 24, 44 \rangle$
$\langle 01, 52, 34, 37 \rangle$							

$n = 35$ :

$\langle \infty, 41, 46, 47 \rangle$	$\langle \infty, 70, 74, 75 \rangle$	$\langle 43, 22, 75, a_1 \rangle$	$\langle 71, 10, 04, a_1 \rangle$	$\langle 62, 43, 15, a_2 \rangle$	$\langle 40, 51, 74, a_2 \rangle$	$\langle 52, 53, 07, a_3 \rangle$	$\langle 21, 50, 24, a_3 \rangle$
$\langle 30, 73, 26, 76 \rangle$	$\langle 62, 53, 47, 55 \rangle$	$\langle 41, 32, 76, 56 \rangle$	$\langle 51, 73, 55, 17 \rangle$	$\langle 13, 03, 67, 04 \rangle$	$\langle 41, 73, 37, 16 \rangle$	$\langle 70, 73, 46, 56 \rangle$	$\langle 42, 03, 25, 16 \rangle$
$\langle 33, 63, 74, 26 \rangle$	$\langle 50, 02, 57, 76 \rangle$	$\langle 51, 63, 67, 34 \rangle$	$\langle 12, 02, 15, 16 \rangle$	$\langle 21, 13, 55, 74 \rangle$	$\langle 10, 70, 35, 05 \rangle$	$\langle 22, 02, 14, 24 \rangle$	$\langle 21, 31, 05, 64 \rangle$
$\langle 70, 11, 37, 67 \rangle$	$\langle 12, 42, 76, 07 \rangle$	$\langle 32, 63, 55, 14 \rangle$	$\langle 40, 12, 17, 27 \rangle$	$\langle 43, 71, 64, 57 \rangle$	$\langle 40, 71, 15, 56 \rangle$	$\langle 30, 42, 74, 14 \rangle$	$\langle 51, 01, 15, 76 \rangle$
$\langle 40, 50, 77, 64 \rangle$	$\langle 10, 01, 75, 46 \rangle$	$\langle 00, 72, 35, 17 \rangle$					

$n = 37$ :

$\langle \infty, 33, 26, 77 \rangle$	$\langle \infty, 00, 04, 55 \rangle$	$\langle 20, 71, 36, a_1 \rangle$	$\langle 32, 13, 55, a_1 \rangle$	$\langle 62, 13, 25, a_2 \rangle$	$\langle 11, 30, 26, a_2 \rangle$	$\langle 71, 70, 46, a_3 \rangle$	$\langle 63, 22, 57, a_3 \rangle$
$\langle 12, 23, 05, a_4 \rangle$	$\langle 00, 21, 54, a_4 \rangle$	$\langle 73, 52, 34, a_5 \rangle$	$\langle 21, 60, 75, a_5 \rangle$	$\langle 01, 51, 06, 76 \rangle$	$\langle 23, 71, 47, 74 \rangle$	$\langle 72, 30, 24, 25 \rangle$	$\langle 73, 23, 75, 54 \rangle$
$\langle 70, 10, 55, 17 \rangle$	$\langle 63, 30, 54, 74 \rangle$	$\langle 32, 52, 74, 06 \rangle$	$\langle 53, 33, 66, 36 \rangle$	$\langle 63, 00, 26, 67 \rangle$	$\langle 22, 10, 46, 47 \rangle$	$\langle 71, 60, 14, 77 \rangle$	$\langle 41, 62, 35, 37 \rangle$
$\langle 42, 61, 25, 45 \rangle$	$\langle 01, 13, 05, 66 \rangle$	$\langle 00, 32, 77, 47 \rangle$	$\langle 73, 11, 47, 57 \rangle$	$\langle 32, 60, 44, 26 \rangle$	$\langle 53, 60, 05, 74 \rangle$	$\langle 22, 30, 35, 07 \rangle$	$\langle 51, 62, 64, 67 \rangle$
$\langle 03, 11, 37, 25 \rangle$	$\langle 12, 70, 56, 76 \rangle$	$\langle 63, 72, 64, 06 \rangle$	$\langle 51, 42, 14, 37 \rangle$	$\langle 01, 11, 74, 35 \rangle$			

$n = 39$ :

$\langle \infty, 11, 17, 16 \rangle$	$\langle \infty, 00, 05, 04 \rangle$	$\langle 20, 21, 34, a_1 \rangle$	$\langle 22, 23, 66, a_1 \rangle$	$\langle 20, 31, 54, a_2 \rangle$	$\langle 32, 43, 56, a_2 \rangle$	$\langle 02, 23, 76, a_3 \rangle$	$\langle 30, 51, 54, a_3 \rangle$
$\langle 30, 61, 14, a_4 \rangle$	$\langle 12, 43, 45, a_4 \rangle$	$\langle 10, 51, 25, a_5 \rangle$	$\langle 42, 03, 44, a_5 \rangle$	$\langle 22, 73, 35, a_6 \rangle$	$\langle 30, 01, 77, a_6 \rangle$	$\langle 02, 63, 14, a_7 \rangle$	$\langle 40, 21, 65, a_7 \rangle$
$\langle 10, 32, 77, 27 \rangle$	$\langle 10, 12, 37, 47 \rangle$	$\langle 00, 10, 65, 45 \rangle$	$\langle 03, 33, 67, 15 \rangle$	$\langle 00, 50, 16, 76 \rangle$	$\langle 11, 72, 46, 57 \rangle$	$\langle 20, 03, 27, 76 \rangle$	$\langle 02, 73, 74, 07 \rangle$
$\langle 10, 13, 07, 67 \rangle$	$\langle 03, 73, 14, 64 \rangle$	$\langle 01, 42, 76, 57 \rangle$	$\langle 01, 02, 75, 64 \rangle$	$\langle 01, 52, 27, 66 \rangle$	$\langle 01, 51, 35, 26 \rangle$	$\langle 20, 11, 56, 15 \rangle$	$\langle 02, 22, 65, 06 \rangle$
$\langle 01, 23, 67, 74 \rangle$	$\langle 10, 30, 04, 16 \rangle$	$\langle 03, 23, 55, 06 \rangle$	$\langle 01, 73, 15, 16 \rangle$	$\langle 01, 33, 25, 37 \rangle$	$\langle 02, 32, 54, 05 \rangle$	$\langle 01, 12, 44, 54 \rangle$	

$n = 41$ :

$\langle \infty, 61, 66, 77 \rangle$	$\langle \infty, 40, 75, 44 \rangle$	$\langle 71, 50, 27, a_1 \rangle$	$\langle 63, 62, 26, a_1 \rangle$	$\langle 31, 20, 54, a_2 \rangle$	$\langle 53, 42, 66, a_2 \rangle$	$\langle 02, 23, 65, a_3 \rangle$	$\langle 00, 01, 14, a_3 \rangle$
$\langle 21, 70, 54, a_4 \rangle$	$\langle 63, 32, 37, a_4 \rangle$	$\langle 73, 32, 54, a_5 \rangle$	$\langle 21, 60, 75, a_5 \rangle$	$\langle 41, 70, 07, a_6 \rangle$	$\langle 22, 73, 55, a_6 \rangle$	$\langle 43, 62, 66, a_7 \rangle$	$\langle 51, 70, 15, a_7 \rangle$
$\langle 00, 71, 54, a_8 \rangle$	$\langle 53, 62, 65, a_8 \rangle$	$\langle 62, 40, 27, a_9 \rangle$	$\langle 33, 01, 16, a_9 \rangle$	$\langle 60, 43, 45, 67 \rangle$	$\langle 43, 23, 76, 37 \rangle$	$\langle 41, 13, 47, 65 \rangle$	$\langle 20, 50, 66, 46 \rangle$
$\langle 11, 42, 77, 56 \rangle$	$\langle 51, 72, 47, 45 \rangle$	$\langle 21, 62, 46, 16 \rangle$	$\langle 70, 32, 34, 26 \rangle$	$\langle 61, 43, 64, 54 \rangle$	$\langle 10, 20, 76, 65 \rangle$	$\langle 60, 33, 55, 17 \rangle$	$\langle 61, 63, 15, 24 \rangle$
$\langle 53, 63, 54, 56 \rangle$	$\langle 22, 02, 15, 45 \rangle$	$\langle 12, 30, 77, 27 \rangle$	$\langle 61, 62, 34, 07 \rangle$	$\langle 31, 01, 66, 15 \rangle$	$\langle 61, 73, 37, 65 \rangle$	$\langle 33, 10, 37, 04 \rangle$	$\langle 12, 60, 04, 66 \rangle$
$\langle 02, 32, 44, 64 \rangle$							

$n = 43$ :

$\langle \infty, 12, 05, 54 \rangle$	$\langle \infty, 01, 07, 76 \rangle$	$\langle 71, 60, 25, a_1 \rangle$	$\langle 43, 12, 04, a_1 \rangle$	$\langle 42, 53, 47, a_2 \rangle$	$\langle 60, 61, 24, a_2 \rangle$	$\langle 62, 03, 76, a_3 \rangle$	$\langle 71, 30, 45, a_3 \rangle$
$\langle 42, 43, 46, a_4 \rangle$	$\langle 61, 70, 64, a_4 \rangle$	$\langle 51, 30, 27, a_5 \rangle$	$\langle 62, 23, 25, a_5 \rangle$	$\langle 42, 23, 65, a_6 \rangle$	$\langle 50, 21, 04, a_6 \rangle$	$\langle 72, 43, 16, a_7 \rangle$	$\langle 31, 50, 75, a_7 \rangle$
$\langle 00, 03, 64, a_8 \rangle$	$\langle 71, 32, 16, a_8 \rangle$	$\langle 52, 50, 24, a_9 \rangle$	$\langle 33, 21, 17, a_9 \rangle$	$\langle 52, 40, 64, a_{10} \rangle$	$\langle 01, 63, 67, a_{10} \rangle$	$\langle 53, 11, 05, a_{11} \rangle$	$\langle 10, 62, 36, a_{11} \rangle$
$\langle 20, 00, 05, 75 \rangle$	$\langle 33, 13, 05, 44 \rangle$	$\langle 33, 50, 16, 67 \rangle$	$\langle 41, 43, 56, 06 \rangle$	$\langle 42, 10, 06, 76 \rangle$	$\langle 42, 20, 36, 67 \rangle$	$\langle 12, 72, 14, 47 \rangle$	$\langle 01, 33, 54, 34 \rangle$
$\langle 31, 02, 77, 55 \rangle$	$\langle 03, 31, 36, 57 \rangle$	$\langle 30, 60, 37, 36 \rangle$	$\langle 01, 23, 74, 05 \rangle$	$\langle 40, 63, 75, 77 \rangle$	$\langle 42, 60, 74, 07 \rangle$	$\langle 12, 11, 75, 25 \rangle$	$\langle 03, 70, 74, 26 \rangle$
$\langle 51, 41, 26, 64 \rangle$	$\langle 03, 11, 47, 27 \rangle$	$\langle 02, 32, 35, 17 \rangle$					

$n = 45$ :

$\langle \infty, 00, 35, 44 \rangle$	$\langle \infty, 61, 17, 56 \rangle$	$\langle 61, 52, 26, a_1 \rangle$	$\langle 60, 33, 64, a_1 \rangle$	$\langle 23, 02, 15, a_2 \rangle$	$\langle 01, 20, 57, a_2 \rangle$	$\langle 70, 71, 77, a_3 \rangle$	$\langle 43, 02, 76, a_3 \rangle$
$\langle 20, 51, 04, a_4 \rangle$	$\langle 53, 42, 37, a_4 \rangle$	$\langle 33, 02, 45, a_5 \rangle$	$\langle 01, 30, 44, a_5 \rangle$	$\langle 73, 02, 47, a_6 \rangle$	$\langle 71, 00, 06, a_6 \rangle$	$\langle 33, 32, 44, a_7 \rangle$	$\langle 20, 31, 75, a_7 \rangle$
$\langle 32, 13, 76, a_8 \rangle$	$\langle 20, 61, 54, a_8 \rangle$	$\langle 53, 51, 05, a_9 \rangle$	$\langle 72, 60, 16, a_9 \rangle$	$\langle 01, 13, 27, a_{10} \rangle$	$\langle 02, 40, 05, a_{10} \rangle$	$\langle 43, 01, 56, a_{11} \rangle$	$\langle 00, 22, 24, a_{11} \rangle$
$\langle 61, 03, 44, a_{12} \rangle$	$\langle 62, 70, 37, a_{12} \rangle$	$\langle 51, 33, 56, a_{13} \rangle$	$\langle 32, 60, 37, a_{13} \rangle$	$\langle 23, 71, 05, 44 \rangle$	$\langle 33, 13, 57, 35 \rangle$	$\langle 30, 51, 25, 17 \rangle$	$\langle 02, 31, 55, 35 \rangle$
$\langle 00, 23, 65, 26 \rangle$	$\langle 41, 73, 16, a_{12} \rangle$	$\langle 70, 62, 07, a_{13} \rangle$	$\langle 31, 43, 44, a_{14} \rangle$	$\langle 42, 10, 06, 76 \rangle$	$\langle 62, 00, 05, 16 \rangle$	$\langle 11, 12, 76, 27 \rangle$	$\langle 41, 62, 76, 66 \rangle$
$\langle 43, 00, 77, 15 \rangle$	$\langle 73, 63, 67, 64 \rangle$	$\langle 61, 51, 47, 45 \rangle$	$\langle 32, 62, 17, 54 \rangle$	$\langle 02, 12, 44, 75 \rangle$			

$n = 47$ :

$\langle \infty, 51, 36, 37 \rangle$	$\langle \infty, 53, 55, 14 \rangle$	$\langle 70, 71, 44, a_1 \rangle$	$\langle 33, 52, 75, a_1 \rangle$	$\langle 63, 60, 45, a_2 \rangle$	$\langle 32, 61, 37, a_2 \rangle$	$\langle 43, 32, 27, a_3 \rangle$	$\langle 00, 31, 14, a_3 \rangle$
$\langle 63, 12, 67, a_4 \rangle$	$\langle 71, 50, 34, a_4 \rangle$	$\langle 23, 62, 16, a_5 \rangle$	$\langle 71, 10, 45, a_5 \rangle$	$\langle 02, 33, 67, a_6 \rangle$	$\langle 70, 61, 76, a_6 \rangle$	$\langle 33, 12, 54, a_7 \rangle$	$\langle 40, 11, 27, a_7 \rangle$
$\langle 10, 21, 17, a_8 \rangle$	$\langle 32, 23, 35, a_8 \rangle$	$\langle 42, 10, 46, a_9 \rangle$	$\langle 61, 63, 55, a_9 \rangle$	$\langle 62, 50, 74, a_{10} \rangle$	$\langle 53, 31, 77, a_{10} \rangle$	$\langle 42, 00, 34, a_{11} \rangle$	$\langle 61, 53, 05, a_{11} \rangle$
$\langle 60, 32, 54, a_{12} \rangle$	$\langle 41, 73, 16, a_{12} \rangle$	$\langle 70, 62, 07, a_{13} \rangle$	$\langle 73, 21, 46, a_{13} \rangle$	$\langle 31, 43, 44, a_{14} \rangle$	$\langle 10, 12, 76, a_{14} \rangle$	$\langle 60, 42, 35, a_{15} \rangle$	$\langle 13, 51, 46, a_{15} \rangle$
$\langle 50, 00, 16, 76 \rangle$	$\langle 00, 73, 04, 37 \rangle$	$\langle 62, 12, 56, 44 \rangle$	$\langle 43, 23, 37, 14 \rangle$	$\langle 70, 00, 47, 05 \rangle$	$\langle 62, 52, 76, 25 \rangle$	$\langle 03, 33, 16, 64 \rangle$	$\langle 10, 33, 55, 07 \rangle$
$\langle 01, 51, 15, 06 \rangle$	$\langle 61, 01, 07, 65 \rangle$	$\langle 72, 12, 27, 25 \rangle$	$\langle 72, 51, 44, 74 \rangle$	$\langle 50, 23, 26, 75 \rangle$	$\langle 32, 41, 06, 77 \rangle$	$\langle 00, 41, 44, 75 \rangle$	

$n = 49$ :

$\langle \infty, 70, 35, 14 \rangle$	$\langle \infty, 31, 47, 06 \rangle$	$\langle \infty, 62, 18, 69 \rangle$	$\langle 63, 42, 77, a_1 \rangle$	$\langle 30, 61, 24, a_1 \rangle$	$\langle 00, 02, 55, 25 \rangle$	$\langle 02, 73, 77, 54 \rangle$	$\langle 13, 10, 58, 57 \rangle$
$\langle 50, 13, 16, 65 \rangle$	$\langle 43, 01, 45, 09 \rangle$	$\langle 22, 23, 66, 59 \rangle$	$\langle 60, 13, 66, 44 \rangle$	$\langle 00, 52, 26, 36 \rangle$	$\langle 10, 20, 79, 59 \rangle$	$\langle 20, 13, 07, 39 \rangle$	$\langle 20, 62, 27, 55 \rangle$
$\langle 53, 40, 68, 77 \rangle$	$\langle 51, 21, 16, 35 \rangle$	$\langle 23, 33, 28, 24 \rangle$	$\langle 63, 51, 38, 17 \rangle$	$\langle 31, 13, 55, 35 \rangle$	$\langle 02, 72, 19, 57 \rangle$	$\langle 60, 01, 36, 48 \rangle$	$\langle 30, 10, 08, 15 \rangle$
$\langle 71, 10, 48, 28 \rangle$	$\langle 41, 22, 54, 38 \rangle$	$\langle 53, 22, 37, 36 \rangle$	$\langle 00, 72, 76, 16 \rangle$	$\langle 41, 33, 19, 09 \rangle$	$\langle 20, 21, 06, 19 \rangle$	$\langle 31, 53, 38, 27 \rangle$	$\langle 31, 03, 54, 19 \rangle$
$\langle 40, 10, 14, 67 \rangle$	$\langle 71, 61, 54, 47 \rangle$	$\langle 50, 02, 04, 59 \rangle$	$\langle 00, 41, 77, 75 \rangle$	$\langle 31, 72, 34, 64 \rangle$	$\langle 42, 13, 54, 09 \rangle$	$\langle 02, 13, 24, 36 \rangle$	$\langle 43, 63, 76, 55 \rangle$
$\langle 60, 42, 68, 24 \rangle$	$\langle 31, 42, 25, 48 \rangle$	$\langle 50, 41, 67, 79 \rangle$	$\langle 12, 32, 37, 45 \rangle$	$\langle 12, 53, 78, 08 \rangle$	$\langle 72, 53, 59, 35 \rangle$	$\langle 41, 12, 15, 68 \rangle$	$\langle 01, 33, 54, 26 \rangle$
$\langle 01, 22, 16, 19 \rangle$							

$n = 51$ :

$\langle \infty, 12, 19, 18 \rangle$	$\langle \infty, 41, 46, 27 \rangle$	$\langle \infty, 40, 34, 35 \rangle$	$\langle 32, 33, 57, a_1 \rangle$	$\langle 60, 61, 36, a_1 \rangle$	$\langle 53, 42, 65, a_2 \rangle$	$\langle 70, 01, 24, a_2 \rangle$	$\langle 70, 21, 04, a_3 \rangle$
$\langle 03, 62, 75, a_3 \rangle$	$\langle 10, 13, 46, 74 \rangle$	$\langle 62, 13, 44, 35 \rangle$	$\langle 30, 21, 16, 45 \rangle$	$\langle 32, 03, 37, 78 \rangle$	$\langle 11, 32, 68, 04 \rangle$	$\langle 60, 52, 27, 06 \rangle$	$\langle 70, 33, 06, 75 \rangle$
$\langle 11, 01, 15, 75 \rangle$	$\langle 20, 33, 38, 16 \rangle$	$\langle 21, 33, 18, 36 \rangle$	$\langle 51, 33, 76, 29 \rangle$	$\langle 41, 72, 06, 58 \rangle$	$\langle 50, 73, 25, 09 \rangle$	$\langle 12, 72, 34, 24 \rangle$	$\langle 20, 10, 08, 64 \rangle$
$\langle 31, 33, 07, 68 \rangle$	$\langle 60, 01, 04, 37 \rangle$	$\langle 10, 60, 69, 45 \rangle$	$\langle 20, 00, 69, 48 \rangle$	$\langle 41, 73, 75, 57 \rangle$	$\langle 31, 42, 05, 66 \rangle$	$\langle 32, 02, 28, 76 \rangle$	$\langle 42, 23, 37, 59 \rangle$
$\langle 71, 42, 35, 18 \rangle$	$\langle 41, 71, 59, 04 \rangle$	$\langle 30, 13, 59, 17 \rangle$	$\langle 42, 03, 09, 16 \rangle$	$\langle 40, 62, 65, 59 \rangle$	$\langle 22, 32, 24, 09 \rangle$	$\langle 30, 32, 36, 67 \rangle$	$\langle 41, 02, 39, 47 \rangle$
$\langle 60, 31, 38, 64 \rangle$	$\langle 23, 53, 24, 46 \rangle$	$\langle 51, 32, 47, 59 \rangle$	$\langle 10, 71, 37, 17 \rangle$	$\langle 52, 43, 68, 37 \rangle$	$\langle 13, 73, 04, 34 \rangle$	$\langle 41, 21, 69, 08 \rangle$	$\langle 40, 13, 57, 39 \rangle$
$\langle 10, 03, 18, 48 \rangle$	$\langle 00, 62, 45, 46 \rangle$	$\langle 03, 13, 65, 69 \rangle$					

$n = 53$ :

$\langle \infty, 50, 35, 44 \rangle$	$\langle \infty, 23, 38, 29 \rangle$	$\langle \infty, 51, 76, 37 \rangle$	$\langle 71, 70, 04, a_1 \rangle$	$\langle 33, 42, 16, a_1 \rangle$	$\langle 10, 21, 44, a_2 \rangle$	$\langle 23, 62, 25, a_2 \rangle$	$\langle 23, 52, 27, a_3 \rangle$
$\langle 00, 21, 45, a_3 \rangle$	$\langle 71, 40, 24, a_4 \rangle$	$\langle 33, 12, 57, a_4 \rangle$	$\langle 21, 60, 75, a_5 \rangle$	$\langle 02, 33, 54, a_5 \rangle$	$\langle 70, 33, 47, 69 \rangle$	$\langle 02, 11, 64, 67 \rangle$	$\langle 22, 50, 75, 29 \rangle$
$\langle 70, 73, 25, 36 \rangle$	$\langle 11, 31, 58, 75 \rangle$	$\langle 03, 02, 58, 69 \rangle$	$\langle 02, 63, 24, 06 \rangle$	$\langle 70, 03, 37, 19 \rangle$	$\langle 20, 30, 38, 74 \rangle$	$\langle 02, 10, 07, 68 \rangle$	$\langle 31, 62, 39, 77 \rangle$
$\langle 40, 52, 49, 77 \rangle$	$\langle 30, 00, 69, 06 \rangle$	$\langle 12, 62, 56, 28 \rangle$	$\langle 12, 50, 48, 54 \rangle$	$\langle 01, 33, 59, 05 \rangle$	$\langle 50, 73, 57, 69 \rangle$	$\langle 41, 03, 04, 59 \rangle$	$\langle 22, 42, 69, 56 \rangle$
$\langle 41, 11, 78, 37 \rangle$	$\langle 41, 63, 76, 29 \rangle$	$\langle 42, 71, 54, 77 \rangle$	$\langle 70, 22, 38, 06 \rangle$	$\langle 11, 22, 39, 25 \rangle$	$\langle 32, 42, 55, 25 \rangle$	$\langle 21, 73, 18, 55 \rangle$	$\langle 70, 41, 39, 16 \rangle$
$\langle 71, 32, 65, 56 \rangle$	$\langle 61, 62, 19, 54 \rangle$	$\langle 10, 03, 77, 15 \rangle$	$\langle 01, 10, 46, 37 \rangle$	$\langle 41, 62, 57, 58 \rangle$	$\langle 60, 41, 46, 56 \rangle$	$\langle 13, 33, 44, 66 \rangle$	$\langle 10, 73, 78, 65 \rangle$
$\langle 51, 43, 28, 46 \rangle$	$\langle 13, 23, 74, 58 \rangle$	$\langle 50, 23, 67, 45 \rangle$	$\langle 00, 22, 24, 48 \rangle$	$\langle 01, 13, 04, 08 \rangle$			

$n = 55$ :

$\langle \infty, 40, 55, 14 \rangle$	$\langle \infty, 33, 09, 38 \rangle$	$\langle \infty, 01, 36, 77 \rangle$	$\langle 63, 62, 05, a_1 \rangle$	$\langle 11, 10, 34, a_1 \rangle$	$\langle 01, 50, 57, a_2 \rangle$	$\langle 33, 22, 25, a_2 \rangle$	$\langle 51, 00, 36, a_3 \rangle$
$\langle 52, 73, 35, a_3 \rangle$	$\langle 73, 22, 17, a_4 \rangle$	$\langle 01, 40, 34, a_4 \rangle$	$\langle 62, 23, 04, a_5 \rangle$	$\langle 70, 01, 75, a_5 \rangle$	$\langle 41, 03, 47, a_6 \rangle$	$\langle 32, 50, 25, a_6 \rangle$	$\langle 03, 20, 16, a_7 \rangle$
$\langle 31, 12, 74, a_7 \rangle$	$\langle 60, 53, 55, 25 \rangle$	$\langle 23, 20, 64, 57 \rangle$	$\langle 13, 43, 75, 14 \rangle$	$\langle 10, 02, 49, 35 \rangle$	$\langle 40, 21, 08, 07 \rangle$	$\langle 10, 33, 26, 24 \rangle$	$\langle 71, 61, 38, 08 \rangle$
$\langle 52, 00, 06, 38 \rangle$	$\langle 31, 63, 19, 14 \rangle$	$\langle 30, 72, 57, 29 \rangle$	$\langle 10, 32, 38, 36 \rangle$	$\langle 03, 50, 79, 19 \rangle$	$\langle 20, 50, 37, 54 \rangle$	$\langle 10, 23, 68, 08 \rangle$	$\langle 20, 73, 79, 77 \rangle$
$\langle 61, 73, 76, 68 \rangle$	$\langle 71, 02, 74, 28 \rangle$	$\langle 01, 61, 45, 17 \rangle$	$\langle 02, 33, 54, 45 \rangle$	$\langle 02, 61, 27, 19 \rangle$	$\langle 00, 21, 19, 35 \rangle$	$\langle 10, 20, 66, 28 \rangle$	$\langle 70, 61, 79, 56 \rangle$
$\langle 11, 33, 19, 08 \rangle$	$\langle 23, 32, 37, 34 \rangle$	$\langle 11, 62, 24, 69 \rangle$	$\langle 61, 11, 15, 35 \rangle$	$\langle 31, 03, 57, 36 \rangle$	$\langle 71, 72, 19, 46 \rangle$	$\langle 72, 40, 37, 28 \rangle$	$\langle 01, 42, 54, 74 \rangle$
$\langle 60, 72, 45, 49 \rangle$	$\langle 02, 22, 37, 46 \rangle$	$\langle 42, 32, 29, 08 \rangle$	$\langle 02, 63, 78, 57 \rangle$	$\langle 41, 72, 06, 66 \rangle$	$\langle 03, 73, 28, 56 \rangle$	$\langle 03, 23, 46, 49 \rangle$	

$n = 57$ :

$\langle \infty, 31, 47, 46 \rangle$	$\langle \infty, 20, 45, 28 \rangle$	$\langle \infty, 43, 44, 29 \rangle$	$\langle 41, 60, 34, a_1 \rangle$	$\langle 73, 02, 36, a_1 \rangle$	$\langle 50, 73, 77, a_2 \rangle$	$\langle 22, 51, 14, a_2 \rangle$	$\langle 32, 53, 46, a_3 \rangle$
$\langle 21, 10, 24, a_3 \rangle$	$\langle 01, 03, 65, a_4 \rangle$	$\langle 22, 40, 64, a_4 \rangle$	$\langle 03, 42, 15, a_5 \rangle$	$\langle 60, 61, 46, a_5 \rangle$	$\langle 62, 13, 36, a_6 \rangle$	$\langle 01, 50, 54, a_6 \rangle$	$\langle 51, 52, 46, a_7 \rangle$
$\langle 13, 50, 67, a_7 \rangle$	$\langle 11, 43, 46, a_8 \rangle$	$\langle 22, 60, 54, a_8 \rangle$	$\langle 12, 60, 24, a_9 \rangle$	$\langle 11, 33, 35, a_9 \rangle$	$\langle 21, 12, 75, 39 \rangle$	$\langle 72, 53, 08, 36 \rangle$	$\langle 21, 33, 54, 28 \rangle$
$\langle 22, 52, 18, 78 \rangle$	$\langle 72, 50, 65, 15 \rangle$	$\langle 10, 20, 78, 67 \rangle$	$\langle 00, 21, 35, 48 \rangle$	$\langle 20, 12, 09, 15 \rangle$	$\langle 52, 00, 65, 76 \rangle$	$\langle 70, 23, 39, 36 \rangle$	$\langle 13, 43, 74, 69 \rangle$
$\langle 13, 51, 45, 38 \rangle$	$\langle 63, 01, 05, 59 \rangle$	$\langle 23, 31, 08, 17 \rangle$	$\langle 01, 53, 37, 18 \rangle$	$\langle 51, 60, 19, 17 \rangle$	$\langle 01, 51, 64, 26 \rangle$	$\langle 03, 23, 14, 37 \rangle$	$\langle 40, 10, 38, 15 \rangle$
$\langle 30, 10, 09, 46 \rangle$	$\langle 12, 32, 16, 27 \rangle$	$\langle 30, 42, 48, 59 \rangle$	$\langle 31, 62, 19, 07 \rangle$	$\langle 63, 73, 78, 35 \rangle$	$\langle 11, 01, 07, 48 \rangle$	$\langle 22, 23, 67, 29 \rangle$	$\langle 50, 11, 37, 56 \rangle$
$\langle 23, 50, 69, 47 \rangle$	$\langle 62, 52, 37, 34 \rangle$	$\langle 62, 51, 48, 15 \rangle$	$\langle 62, 41, 64, 39 \rangle$	$\langle 02, 41, 69, 49 \rangle$	$\langle 30, 01, 39, 06 \rangle$	$\langle 00, 02, 07, 38 \rangle$	$\langle 63, 70, 16, 24 \rangle$
$\langle 02, 53, 24, 45 \rangle$							

$n = 59$ :

$\langle \infty, 30, 77, 56 \rangle$	$\langle \infty, 21, 49, 64 \rangle$	$\langle \infty, 33, 65, 18 \rangle$	$\langle 32, 33, 46, a_1 \rangle$	$\langle 40, 31, 44, a_1 \rangle$	$\langle 32, 30, 24, a_2 \rangle$	$\langle 61, 13, 66, a_2 \rangle$	$\langle 63, 11, 17, a_3 \rangle$
$\langle 22, 70, 76, a_3 \rangle$	$\langle 23, 32, 67, a_4 \rangle$	$\langle 20, 71, 68, a_4 \rangle$	$\langle 41, 23, 77, a_5 \rangle$	$\langle 02, 70, 24, a_5 \rangle$	$\langle 63, 20, 56, a_6 \rangle$	$\langle 71, 02, 25, a_6 \rangle$	$\langle 11, 72, 25, a_7 \rangle$
$\langle 33, 50, 66, a_7 \rangle$	$\langle 13, 71, 54, a_8 \rangle$	$\langle 60, 42, 67, a_8 \rangle$	$\langle 42, 50, 14, a_9 \rangle$	$\langle 21, 63, 45, a_9 \rangle$	$\langle 30, 23, 27, a_{10} \rangle$	$\langle 21, 12, 14, a_{10} \rangle$	$\langle 63, 32, 07, a_{11} \rangle$
$\langle 21, 70, 46, a_{11} \rangle$	$\langle 00, 70, 38, 55 \rangle$	$\langle 30, 43, 45, 38 \rangle$	$\langle 11, 42, 55, 18 \rangle$	$\langle 41, 30, 59, 25 \rangle$	$\langle 60, 01, 77, 78 \rangle$	$\langle 10, 70, 15, 79 \rangle$	$\langle 10, 62, 24, 49 \rangle$
$\langle 52, 72, 58, 49 \rangle$	$\langle 02, 53, 67, 47 \rangle$	$\langle 10, 51, 77, 45 \rangle$	$\langle 62, 52, 09, 56 \rangle$	$\langle 11, 23, 49, 48 \rangle$	$\langle 23, 53, 69, 45 \rangle$	$\langle 40, 21, 09, 67 \rangle$	$\langle 12, 70, 27, 48 \rangle$
$\langle 43, 40, 24, 39 \rangle$	$\langle 41, 61, 14, 28 \rangle$	$\langle 03, 30, 26, 18 \rangle$	$\langle 01, 03, 55, 46 \rangle$	$\langle 22, 01, 18, 56 \rangle$	$\langle 22, 43, 05, 34 \rangle$	$\langle 22, 61, 65, 04 \rangle$	$\langle 12, 53, 36, 58 \rangle$
$\langle 12, 73, 28, 76 \rangle$	$\langle 13, 23, 24, 68 \rangle$	$\langle 42, 53, 37, 74 \rangle$	$\langle 50, 03, 69, 15 \rangle$	$\langle 02, 52, 19, 55 \rangle$	$\langle 13, 33, 69, 64 \rangle$	$\langle 10, 60, 08, 34 \rangle$	$\langle 71, 41, 39, 57 \rangle$
$\langle 71, 72, 18, 79 \rangle$	$\langle 00, 01, 59, 66 \rangle$	$\langle 01, 52, 16, 57 \rangle$					

$n = 61$ :

$\langle \infty, 70, 34, 05 \rangle$	$\langle \infty, 02, 06, 67 \rangle$	$\langle \infty, 53, 59, 18 \rangle$	$\langle 13, 10, 74, a_1 \rangle$	$\langle 01, 32, 55, a_1 \rangle$	$\langle 01, 23, 14, a_2 \rangle$	$\langle 42, 00, 16, a_2 \rangle$	$\langle 31, 22, 74, a_3 \rangle$
$\langle 10, 73, 75, a_3 \rangle$	$\langle 21, 22, 17, a_4 \rangle$	$\langle 30, 63, 34, a_4 \rangle$	$\langle 71, 33, 77, a_5 \rangle$	$\langle 10, 72, 36, a_5 \rangle$	$\langle 33, 31, 25, a_6 \rangle$	$\langle 50, 22, 27, a_6 \rangle$	$\langle 51, 32, 07, a_7 \rangle$
$\langle 33, 70, 75, a_7 \rangle$	$\langle 33, 12, 57, a_8 \rangle$	$\langle 21, 10, 16, a_8 \rangle$	$\langle 70, 43, 17, a_9 \rangle$	$\langle 62, 41, 74, a_9 \rangle$	$\langle 51, 50, 74, a_{10} \rangle$	$\langle 12, 13, 27, a_{10} \rangle$	$\langle 33, 41, 16, a_{11} \rangle$
$\langle 42, 10, 55, a_{11} \rangle$	$\langle 60, 51, 15, a_{12} \rangle$	$\langle 33, 42, 56, a_{12} \rangle$	$\langle 30, 23, 27, a_{13} \rangle$	$\langle 62, 11, 64, a_{13} \rangle$	$\langle 33, 01, 18, 68 \rangle$	$\langle 12, 32, 15, 28 \rangle$	$\langle 70, 31, 07, 29 \rangle$
$\langle 43, 71, 74, 37 \rangle$	$\langle 20, 01, 16, 74 \rangle$	$\langle 32, 73, 08, 57 \rangle$	$\langle 41, 52, 69, 38 \rangle$	$\langle 03, 63, 24, 15 \rangle$	$\langle 20, 32, 66, 14 \rangle$	$\langle 52, 22, 15, 09 \rangle$	$\langle 51, 61, 28, 05 \rangle$
$\langle 42, 53, 69, 39 \rangle$	$\langle 21, 03, 58, 37 \rangle$	$\langle 61, 31, 45, 69 \rangle$	$\langle 40, 60, 18, 27 \rangle$	$\langle 60, 30, 55, 59 \rangle$	$\langle 31, 00, 66, 49 \rangle$	$\langle 02, 12, 59, 44 \rangle$	$\langle 11, 52, 74, 76 \rangle$
$\langle 50, 71, 59, 57 \rangle$	$\langle 03, 53, 79, 06 \rangle$	$\langle 01, 13, 06, 59 \rangle$	$\langle 60, 31, 79, 35 \rangle$	$\langle 21, 01, 28, 46 \rangle$	$\langle 50, 73, 04, 29 \rangle$	$\langle 30, 40, 48, 28 \rangle$	$\langle 60, 73, 49, 74 \rangle$
$\langle 02, 63, 76, 08 \rangle$	$\langle 50, 42, 78, 26 \rangle$	$\langle 40, 42, 77, 08 \rangle$	$\langle 33, 02, 55, 28 \rangle$	$\langle 03, 13, 65, 56 \rangle$			

$n = 63$ :

$(\infty, 00, 15, 04)$	$(\infty, 12, 38, 26)$	$(\infty, 43, 07, 09)$	$(62, 10, 56, a_1)$	$(71, 43, 64, a_1)$	$(60, 71, 65, a_2)$	$(12, 23, 36, a_2)$	$(00, 23, 75, a_3)$
$(01, 02, 67, a_3)$	$(12, 60, 56, a_4)$	$(31, 33, 74, a_4)$	$(11, 60, 46, a_5)$	$(42, 13, 45, a_5)$	$(60, 61, 34, a_6)$	$(12, 73, 57, a_6)$	$(61, 03, 07, a_7)$
$(32, 70, 34, a_7)$	$(30, 03, 14, a_8)$	$(61, 72, 27, a_8)$	$(31, 12, 27, a_9)$	$(50, 63, 44, a_9)$	$(72, 21, 76, a_{10})$	$(50, 33, 47, a_{10})$	$(42, 03, 74, a_{11})$
$(21, 40, 46, a_{11})$	$(10, 72, 55, a_{12})$	$(51, 03, 34, a_{12})$	$(41, 70, 77, a_{13})$	$(53, 32, 16, a_{13})$	$(11, 73, 26, a_{14})$	$(20, 12, 37, a_{14})$	$(71, 50, 35, a_{15})$
$(73, 02, 56, a_{15})$	$(21, 63, 08, 55)$	$(43, 23, 65, 58)$	$(10, 12, 45, 79)$	$(30, 71, 07, a_{18})$	$(02, 03, 59, 54)$	$(01, 72, 48, 14)$	$(10, 43, 37, 44)$
$(62, 72, 67, 78)$	$(03, 73, 05, 58)$	$(20, 11, 19, 75)$	$(10, 32, 66, 28)$	$(41, 53, 09, 58)$	$(72, 22, 04, 19)$	$(61, 22, 39, 14)$	$(01, 22, 58, 15)$
$(00, 12, 25, 19)$	$(12, 72, 35, 58)$	$(60, 40, 09, 27)$	$(40, 30, 54, 28)$	$(30, 00, 58, 38)$	$(10, 53, 46, 29)$	$(42, 73, 17, 09)$	$(61, 31, 37, 58)$
$(10, 13, 19, 47)$	$(21, 13, 75, 39)$	$(30, 23, 09, 46)$	$(03, 53, 48, 56)$	$(12, 61, 49, 65)$	$(01, 71, 76, 29)$	$(01, 21, 24, 66)$	

$n = 65$ :

$\langle \infty, 13, 610, 16 \rangle$	$\langle \infty, 50, 68, 27 \rangle$	$\langle \infty, 31, 05, 411 \rangle$	$\langle \infty, 32, 09, 34 \rangle$	$\langle 42, 71, 15, a_1 \rangle$	$\langle 30, 43, 26, a_1 \rangle$	$\langle 20, 11, 210, 46 \rangle$	$\langle 21, 51, 19, 54 \rangle$
$\langle 02, 12, 15, 211 \rangle$	$\langle 73, 72, 59, 35 \rangle$	$\langle 11, 32, 48, 74 \rangle$	$\langle 31, 63, 210, 45 \rangle$	$\langle 23, 33, 28, 44 \rangle$	$\langle 30, 11, 47, 410 \rangle$	$\langle 51, 33, 26, 18 \rangle$	$\langle 00, 72, 610, 47 \rangle$
$\langle 22, 42, 16, 210 \rangle$	$\langle 50, 30, 511, 65 \rangle$	$\langle 42, 70, 411, 211 \rangle$	$\langle 01, 61, 06, 411 \rangle$	$\langle 11, 60, 411, 57 \rangle$	$\langle 01, 13, 511, 211 \rangle$	$\langle 31, 53, 07, 710 \rangle$	$\langle 41, 72, 19, 25 \rangle$
$\langle 30, 00, 05, 210 \rangle$	$\langle 10, 32, 18, 08 \rangle$	$\langle 52, 33, 24, 58 \rangle$	$\langle 40, 11, 76, 18 \rangle$	$\langle 30, 40, 59, 68 \rangle$	$\langle 42, 72, 54, 010 \rangle$	$\langle 33, 12, 47, 611 \rangle$	$\langle 22, 13, 111, 39 \rangle$
$\langle 40, 02, 26, 16 \rangle$	$\langle 00, 41, 27, 59 \rangle$	$\langle 01, 42, 610, 07 \rangle$	$\langle 21, 23, 38, 210 \rangle$	$\langle 22, 31, 27, 19 \rangle$	$\langle 41, 20, 34, 54 \rangle$	$\langle 20, 32, 57, 27 \rangle$	$\langle 02, 43, 310, 75 \rangle$
$\langle 31, 41, 311, 28 \rangle$	$\langle 01, 73, 75, 210 \rangle$	$\langle 01, 62, 16, 29 \rangle$	$\langle 03, 10, 56, 58 \rangle$	$\langle 61, 72, 38, 24 \rangle$	$\langle 21, 63, 55, 29 \rangle$	$\langle 60, 13, 210, 49 \rangle$	$\langle 20, 03, 611, 15 \rangle$
$\langle 00, 62, 310, 45 \rangle$	$\langle 31, 32, 47, 58 \rangle$	$\langle 33, 10, 09, 74 \rangle$	$\langle 03, 33, 34, 27 \rangle$	$\langle 10, 42, 78, 49 \rangle$	$\langle 10, 21, 16, 59 \rangle$	$\langle 52, 63, 44, 011 \rangle$	$\langle 30, 73, 15, 411 \rangle$
$\langle 02, 53, 411, 25 \rangle$	$\langle 00, 02, 67, 24 \rangle$	$\langle 30, 31, 04, 010 \rangle$	$\langle 13, 41, 64, 67 \rangle$	$\langle 13, 33, 29, 56 \rangle$	$\langle 00, 03, 16, 09 \rangle$	$\langle 02, 33, 66, 78 \rangle$	$\langle 00, 40, 04, 44 \rangle^s$
$\langle 01, 41, 05, 45 \rangle^s$	$\langle 02, 42, 06, 46 \rangle^s$	$\langle 03, 43, 07, 47 \rangle^s$					

Note that each of the codewords marked  $s$  only generates four codewords. ■

**Lemma 8.7:**  $T(2, 37; 2, n; 6) = 9n$  for each odd  $n$  and  $37 \leq n \leq 73$ .

*Proof:* Let  $X_1 = (\mathbb{Z}_9 \times \{0, 1, 2, 3\}) \cup \{\infty\}$ . For  $37 \leq n \leq 53$ , let  $X_2 = (\mathbb{Z}_9 \times \{4, 5, 6, 7\}) \cup (\{a\} \times \{1, \dots, n-36\})$ ; for  $55 \leq n \leq 71$ , let  $X_2 = (\mathbb{Z}_9 \times \{4, 5, \dots, 9\}) \cup (\{a\} \times \{1, \dots, n-54\})$ ; for  $n = 73$ , let  $X_2 = (\mathbb{Z}_9 \times \{4, 5, \dots, 11\}) \cup (\{a\} \times \{1\})$ . Denote  $X = X_1 \cup X_2$ . The desired codes of size  $9n$  are constructed on  $\mathbb{Z}_2^X$  and the base codewords are listed as follows.

$n = 37$ :

$\langle \infty, 40, 44, 45 \rangle$	$\langle \infty, 01, 07, 06 \rangle$	$\langle 42, 43, 45, a_1 \rangle$	$\langle 50, 81, 64, a_1 \rangle$	$\langle 10, 71, 56, 67 \rangle$	$\langle 12, 03, 47, 64 \rangle$	$\langle 03, 33, 27, 55 \rangle$	$\langle 10, 81, 37, 07 \rangle$
$\langle 01, 81, 84, 24 \rangle$	$\langle 02, 72, 07, 14 \rangle$	$\langle 20, 41, 05, 56 \rangle$	$\langle 23, 63, 35, 36 \rangle$	$\langle 32, 43, 85, 04 \rangle$	$\langle 22, 63, 37, 77 \rangle$	$\langle 21, 22, 45, 64 \rangle$	$\langle 11, 31, 15, 75 \rangle$
$\langle 12, 02, 04, 45 \rangle$	$\langle 10, 30, 04, 25 \rangle$	$\langle 20, 71, 16, 57 \rangle$	$\langle 03, 13, 06, 84 \rangle$	$\langle 10, 21, 55, 84 \rangle$	$\langle 32, 63, 26, 05 \rangle$	$\langle 00, 50, 56, 66 \rangle$	$\langle 20, 23, 06, 46 \rangle$
$\langle 02, 32, 15, 06 \rangle$	$\langle 11, 42, 37, 76 \rangle$	$\langle 12, 52, 36, 34 \rangle$	$\langle 22, 43, 07, 76 \rangle$	$\langle 20, 50, 75, 85 \rangle$	$\langle 20, 22, 67, 87 \rangle$	$\langle 01, 33, 15, 54 \rangle$	$\langle 01, 72, 26, 86 \rangle$
$\langle 00, 80, 44, 24 \rangle$	$\langle 00, 41, 17, 07 \rangle$	$\langle 03, 73, 14, 77 \rangle$	$\langle 41, 43, 77, 35 \rangle$	$\langle 01, 13, 14, 56 \rangle$			

$n = 39$ :

$\langle \infty, 00, 04, 05 \rangle$	$\langle \infty, 01, 06, 07 \rangle$	$\langle 12, 13, 15, a_1 \rangle$	$\langle 30, 31, 44, a_1 \rangle$	$\langle 30, 41, 64, a_2 \rangle$	$\langle 22, 33, 37, a_2 \rangle$	$\langle 12, 63, 25, a_3 \rangle$	$\langle 30, 51, 54, a_3 \rangle$
$\langle 00, 10, 74, 54 \rangle$	$\langle 00, 53, 47, 77 \rangle$	$\langle 03, 43, 77, 35 \rangle$	$\langle 03, 83, 84, 65 \rangle$	$\langle 10, 71, 27, 17 \rangle$	$\langle 12, 73, 47, 76 \rangle$	$\langle 01, 31, 86, 66 \rangle$	$\langle 00, 60, 66, 35 \rangle$
$\langle 03, 33, 66, 14 \rangle$	$\langle 01, 71, 75, 65 \rangle$	$\langle 03, 23, 76, 54 \rangle$	$\langle 02, 43, 07, 56 \rangle$	$\langle 12, 43, 64, 87 \rangle$	$\langle 00, 43, 84, 55 \rangle$	$\langle 02, 22, 26, 75 \rangle$	$\langle 02, 52, 74, 86 \rangle$
$\langle 12, 83, 07, 54 \rangle$	$\langle 11, 12, 04, 55 \rangle$	$\langle 10, 33, 56, 26 \rangle$	$\langle 00, 50, 76, 86 \rangle$	$\langle 02, 32, 46, 64 \rangle$	$\langle 10, 81, 67, 47 \rangle$	$\langle 10, 12, 37, 77 \rangle$	$\langle 00, 70, 25, 85 \rangle$
$\langle 01, 11, 74, 44 \rangle$	$\langle 02, 82, 04, 25 \rangle$	$\langle 00, 41, 56, 75 \rangle$	$\langle 01, 51, 76, 67 \rangle$	$\langle 01, 42, 15, 87 \rangle$	$\langle 11, 72, 65, 37 \rangle$	$\langle 01, 03, 25, 46 \rangle$	

$n = 41$ :

$\langle \infty, 00, 04, 05 \rangle$	$\langle \infty, 01, 06, 07 \rangle$	$\langle 02, 03, 55, a_1 \rangle$	$\langle 10, 11, 24, a_1 \rangle$	$\langle 00, 11, 34, a_2 \rangle$	$\langle 02, 13, 25, a_2 \rangle$	$\langle 02, 23, 15, a_3 \rangle$	$\langle 00, 71, 17, a_3 \rangle$
$\langle 00, 31, 64, a_4 \rangle$	$\langle 02, 53, 17, a_4 \rangle$	$\langle 02, 73, 16, a_5 \rangle$	$\langle 40, 81, 34, a_5 \rangle$	$\langle 01, 21, 86, 35 \rangle$	$\langle 00, 81, 37, 74 \rangle$	$\langle 00, 30, 06, 16 \rangle$	$\langle 02, 33, 37, 05 \rangle$
$\langle 01, 12, 87, 36 \rangle$	$\langle 01, 82, 77, 17 \rangle$	$\langle 02, 42, 36, 35 \rangle$	$\langle 10, 12, 66, 57 \rangle$	$\langle 03, 43, 16, 27 \rangle$	$\langle 10, 22, 77, 87 \rangle$	$\langle 01, 11, 64, 05 \rangle$	$\langle 11, 13, 56, 77 \rangle$
$\langle 01, 02, 65, 74 \rangle$	$\langle 03, 23, 06, 64 \rangle$	$\langle 01, 73, 25, 27 \rangle$	$\langle 01, 31, 56, 75 \rangle$	$\langle 00, 61, 27, 46 \rangle$	$\langle 02, 12, 04, 24 \rangle$	$\langle 00, 40, 07, 36 \rangle$	$\langle 02, 32, 06, 76 \rangle$
$\langle 02, 72, 34, 44 \rangle$	$\langle 02, 63, 45, 07 \rangle$	$\langle 03, 13, 04, 34 \rangle$	$\langle 03, 33, 26, 14 \rangle$	$\langle 00, 20, 65, 55 \rangle$	$\langle 00, 63, 85, 26 \rangle$	$\langle 00, 73, 75, 87 \rangle$	$\langle 00, 51, 54, 15 \rangle$
$\langle 00, 83, 44, 25 \rangle$							

$n = 43$ :

$\langle \infty, 31, 17, 86 \rangle$	$\langle \infty, 20, 54, 35 \rangle$	$\langle 30, 31, 44, a_1 \rangle$	$\langle 13, 12, 15, a_1 \rangle$	$\langle 03, 82, 26, a_2 \rangle$	$\langle 60, 71, 47, a_2 \rangle$	$\langle 42, 63, 55, a_3 \rangle$	$\langle 01, 70, 44, a_3 \rangle$
$\langle 52, 33, 87, a_4 \rangle$	$\langle 60, 01, 54, a_4 \rangle$	$\langle 80, 41, 85, a_5 \rangle$	$\langle 22, 63, 17, a_5 \rangle$	$\langle 42, 03, 75, a_6 \rangle$	$\langle 50, 31, 46, a_6 \rangle$	$\langle 80, 51, 47, a_7 \rangle$	$\langle 42, 13, 44, a_7 \rangle$
$\langle 11, 43, 86, 46 \rangle$	$\langle 13, 53, 34, 24 \rangle$	$\langle 11, 21, 15, 37 \rangle$	$\langle 11, 52, 57, 17 \rangle$	$\langle 13, 83, 26, 35 \rangle$	$\langle 10, 30, 05, 84 \rangle$	$\langle 22, 70, 04, 07 \rangle$	$\langle 31, 53, 04, 87 \rangle$
$\langle 13, 31, 76, 64 \rangle$	$\langle 20, 50, 36, 26 \rangle$	$\langle 00, 22, 87, 67 \rangle$	$\langle 12, 72, 86, 24 \rangle$	$\langle 10, 00, 35, 55 \rangle$	$\langle 42, 11, 34, 04 \rangle$	$\langle 13, 27, 14, 87 \rangle$	$\langle 12, 02, 34, 65 \rangle$
$\langle 11, 12, 36, 85 \rangle$	$\langle 11, 23, 75, 16 \rangle$	$\langle 00, 73, 36, 07 \rangle$	$\langle 21, 32, 55, 57 \rangle$	$\langle 30, 62, 34, 56 \rangle$	$\langle 12, 62, 16, 55 \rangle$	$\langle 10, 53, 57, 47 \rangle$	$\langle 01, 83, 55, 25 \rangle$
$\langle 30, 03, 76, 15 \rangle$	$\langle 50, 52, 16, 67 \rangle$	$\langle 01, 21, 04, 86 \rangle$					

$n = 45$ :

$\langle \infty, 60, 45, 54 \rangle$	$\langle \infty, 61, 67, 56 \rangle$	$\langle 40, 51, 04, a_1 \rangle$	$\langle 13, 12, 77, a_1 \rangle$	$\langle 12, 23, 67, a_2 \rangle$	$\langle 11, 80, 85, a_2 \rangle$	$\langle 80, 81, 04, a_3 \rangle$	$\langle 73, 52, 75, a_3 \rangle$
$\langle 00, 31, 57, a_4 \rangle$	$\langle 32, 63, 85, a_4 \rangle$	$\langle 42, 83, 67, a_5 \rangle$	$\langle 40, 21, 44, a_5 \rangle$	$\langle 12, 63, 76, a_6 \rangle$	$\langle 30, 71, 64, a_6 \rangle$	$\langle 63, 02, 34, a_7 \rangle$	$\langle 30, 01, 45, a_7 \rangle$
$\langle 00, 32, 35, a_8 \rangle$	$\langle 01, 63, 46, a_8 \rangle$	$\langle 32, 23, 44, a_9 \rangle$	$\langle 11, 20, 67, a_9 \rangle$	$\langle 71, 62, 17, 67 \rangle$	$\langle 42, 72, 86, 55 \rangle$	$\langle 50, 02, 27, 37 \rangle$	$\langle 12, 32, 47, 74 \rangle$
$\langle 73, 61, 85, 25 \rangle$	$\langle 81, 42, 46, 37 \rangle$	$\langle 71, 22, 85, 44 \rangle$	$\langle 80, 62, 64, 06 \rangle$	$\langle 31, 81, 66, 06 \rangle$	$\langle 70, 72, 66, 65 \rangle$	$\langle 81, 21, 24, 85 \rangle$	$\langle 81, 13, 67, 07 \rangle$
$\langle 41, 03, 46, 17 \rangle$	$\langle 23, 80, 15, 54 \rangle$	$\langle 12, 00, 36, 87 \rangle$	$\langle 02, 01, 74, 54 \rangle$	$\langle 33, 63, 24, 14 \rangle$	$\langle 60, 73, 46, 77 \rangle$	$\langle 12, 60, 66, 86 \rangle$	$\langle 53, 73, 16, 35 \rangle$
$\langle 63, 53, 64, 56 \rangle$	$\langle 20, 40, 86, 64 \rangle$	$\langle 60, 82, 25, 35 \rangle$	$\langle 61, 63, 86, 05 \rangle$	$\langle 00, 73, 45, 07 \rangle$			

$n = 47$ :

$\langle \infty, 50, 75, 54 \rangle$	$\langle \infty, 41, 77, 36 \rangle$	$\langle 40, 41, 54, a_1 \rangle$	$\langle 83, 42, 26, a_1 \rangle$	$\langle 52, 63, 75, a_2 \rangle$	$\langle 51, 40, 16, a_2 \rangle$	$\langle 13, 82, 65, a_3 \rangle$	$\langle 80, 11, 46, a_3 \rangle$
$\langle 51, 20, 84, a_4 \rangle$	$\langle 33, 02, 55, a_4 \rangle$	$\langle 02, 03, 05, a_5 \rangle$	$\langle 40, 81, 74, a_5 \rangle$	$\langle 71, 20, 56, a_6 \rangle$	$\langle 63, 12, 45, a_6 \rangle$	$\langle 33, 62, 07, a_7 \rangle$	$\langle 20, 81, 35, a_7 \rangle$
$\langle 02, 73, 27, a_8 \rangle$	$\langle 21, 40, 04, a_8 \rangle$	$\langle 13, 22, 67, a_9 \rangle$	$\langle 81, 00, 85, a_9 \rangle$	$\langle 42, 60, 17, a_{10} \rangle$	$\langle 41, 23, 55, a_{10} \rangle$	$\langle 32, 60, 45, a_{11} \rangle$	$\langle 03, 71, 44, a_{11} \rangle$
$\langle 52, 21, 67, 45 \rangle$	$\langle 30, 60, 37, 14 \rangle$	$\langle 23, 83, 07, 04 \rangle$	$\langle 30, 50, 85, 05 \rangle$	$\langle 40, 00, 37, 27 \rangle$	$\langle 41, 73, 25, 86 \rangle$	$\langle 22, 32, 64, 44 \rangle$	$\langle 41, 21, 75, 44 \rangle$
$\langle 11, 21, 37, 36 \rangle$	$\langle 33, 13, 76, 36 \rangle$	$\langle 33, 83, 54, 27 \rangle$	$\langle 61, 73, 14, 36 \rangle$	$\langle 42, 22, 66, 04 \rangle$	$\langle 51, 02, 56, 45 \rangle$	$\langle 43, 33, 34, 26 \rangle$	$\langle 11, 63, 87, 67 \rangle$
$\langle 40, 82, 56, 26 \rangle$	$\langle 51, 53, 25, 14 \rangle$	$\langle 50, 42, 56, 46 \rangle$	$\langle 12, 50, 17, 06 \rangle$	$\langle 60, 02, 77, 65 \rangle$	$\langle 42, 20, 14, 44 \rangle$	$\langle 01, 12, 07, 67 \rangle$	

$n = 49$ :

$\langle \infty, 30, 15, 14 \rangle$	$\langle \infty, 31, 76, 07 \rangle$	$\langle 42, 43, 85, a_1 \rangle$	$\langle 41, 40, 54, a_1 \rangle$	$\langle 43, 32, 55, a_2 \rangle$	$\langle 31, 20, 54, a_2 \rangle$	$\langle 01, 70, 04, a_3 \rangle$	$\langle 62, 83, 75, a_3 \rangle$
$\langle 13, 72, 35, a_4 \rangle$	$\langle 11, 70, 06, a_4 \rangle$	$\langle 33, 82, 65, a_5 \rangle$	$\langle 80, 31, 57, a_5 \rangle$	$\langle 60, 21, 14, a_6 \rangle$	$\langle 13, 52, 67, a_6 \rangle$	$\langle 43, 72, 74, a_7 \rangle$	$\langle 81, 20, 35, a_7 \rangle$
$\langle 33, 52, 17, a_8 \rangle$	$\langle 11, 30, 84, a_8 \rangle$	$\langle 81, 00, 25, a_9 \rangle$	$\langle 23, 32, 44, a_9 \rangle$	$\langle 51, 53, 35, a_{10} \rangle$	$\langle 12, 20, 77, a_{10} \rangle$	$\langle 03, 41, 37, a_{11} \rangle$	$\langle 12, 10, 76, a_{11} \rangle$
$\langle 40, 72, 27, a_{12} \rangle$	$\langle 61, 53, 24, a_{12} \rangle$	$\langle 60, 82, 06, a_{13} \rangle$	$\langle 61, 03, 27, a_{13} \rangle$	$\langle 22, 52, 25, 56 \rangle$	$\langle 41, 53, 55, 26 \rangle$	$\langle 10, 70, 86, 66 \rangle$	$\langle 41, 62, 46, 87 \rangle$
$\langle 21, 32, 86, 54 \rangle$	$\langle 10, 80, 55, 45 \rangle$	$\langle 13, 23, 75, 14 \rangle$	$\langle 40, 52, 57, 46 \rangle$	$\langle 02, 82, 44, 74 \rangle$	$\langle 23, 71, 17, 16 \rangle$	$\langle 10, 83, 07, 37 \rangle$	$\langle 51, 32, 04, 76 \rangle$
$\langle 23, 53, 76, 64 \rangle$	$\langle 31, 11, 17, 05 \rangle$	$\langle 11, 61, 26, 65 \rangle$	$\langle 33, 53, 66, 36 \rangle$	$\langle 30, 43, 04, 24 \rangle$	$\langle 60, 63, 16, 67 \rangle$	$\langle 81, 22, 07, 54 \rangle$	$\langle 80, 00, 85, 37 \rangle$
$\langle 02, 42, 35, 37 \rangle$							

$n = 51$ :

$\langle \infty, 11, 16, 17 \rangle$	$\langle \infty, 10, 34, 05 \rangle$	$\langle 32, 33, 35, a_1 \rangle$	$\langle 10, 11, 24, a_1 \rangle$	$\langle 00, 11, 34, a_2 \rangle$	$\langle 12, 23, 35, a_2 \rangle$	$\langle 12, 33, 25, a_3 \rangle$	$\langle 20, 41, 77, a_3 \rangle$
$\langle 22, 53, 56, a_4 \rangle$	$\langle 20, 51, 84, a_4 \rangle$	$\langle 30, 71, 24, a_5 \rangle$	$\langle 22, 63, 05, a_5 \rangle$	$\langle 22, 73, 55, a_6 \rangle$	$\langle 50, 11, 04, a_6 \rangle$	$\langle 40, 11, 55, a_7 \rangle$	$\langle 12, 73, 07, a_7 \rangle$
$\langle 32, 13, 75, a_8 \rangle$	$\langle 30, 11, 84, a_8 \rangle$	$\langle 12, 03, 24, a_9 \rangle$	$\langle 10, 01, 35, a_9 \rangle$	$\langle 21, 23, 87, a_{10} \rangle$	$\langle 20, 22, 04, a_{10} \rangle$	$\langle 01, 83, 54, a_{11} \rangle$	$\langle 10, 22, 85, a_{11} \rangle$
$\langle 21, 63, 24, a_{12} \rangle$	$\langle 10, 32, 16, a_{12} \rangle$	$\langle 11, 43, 65, a_{13} \rangle$	$\langle 10, 42, 36, a_{13} \rangle$	$\langle 21, 43, 16, a_{14} \rangle$	$\langle 10, 52, 45, a_{14} \rangle$	$\langle 01, 53, 64, a_{15} \rangle$	$\langle 20, 72, 36, a_{15} \rangle$
$\langle 01, 13, 86, 65 \rangle$	$\langle 03, 13, 04, 56 \rangle$	$\langle 03, 63, 17, 86 \rangle$	$\langle 02, 22, 24, 54 \rangle$	$\langle 00, 72, 47, 87 \rangle$	$\langle 01, 62, 56, 37 \rangle$	$\langle 01, 62, 25, 66 \rangle$	$\langle 03, 23, 07, 36 \rangle$
$\langle 02, 42, 84, 27 \rangle$	$\langle 03, 43, 37, 74 \rangle$	$\langle 20, 82, 06, 47 \rangle$	$\langle 11, 62, 86, 07 \rangle$	$\langle 11, 22, 66, 27 \rangle$	$\langle 01, 11, 05, 75 \rangle$	$\langle 00, 30, 17, 07 \rangle$	$\langle 00, 10, 46, 55 \rangle$
$\langle 01, 31, 77, 57 \rangle$	$\langle 01, 71, 26, 16 \rangle$	$\langle 00, 53, 04, 05 \rangle$					

$n = 53$ :

$(\infty, 31, 36, 37)$	$(\infty, 40, 44, 45)$	$(01, 00, 14, a_1)$	$(32, 33, 45, a_1)$	$(52, 63, 16, a_2)$	$(01, 80, 24, a_2)$	$(41, 20, 44, a_3)$	$(52, 73, 75, a_3)$
$(80, 21, 54, a_4)$	$(72, 13, 65, a_4)$	$(40, 81, 34, a_5)$	$(52, 03, 27, a_5)$	$(11, 50, 04, a_6)$	$(52, 13, 85, a_6)$	$(21, 50, 65, a_7)$	$(42, 13, 44, a_7)$
$(62, 43, 66, a_8)$	$(61, 80, 44, a_8)$	$(81, 00, 25, a_9)$	$(72, 63, 14, a_9)$	$(81, 83, 55, a_{10})$	$(42, 40, 24, a_{10})$	$(81, 03, 67, a_{11})$	$(60, 52, 05, a_{11})$
$(22, 80, 77, a_{12})$	$(61, 03, 24, a_{12})$	$(61, 43, 77, a_{13})$	$(12, 60, 86, a_{13})$	$(70, 02, 87, a_{14})$	$(13, 81, 06, a_{14})$	$(50, 12, 27, a_{15})$	$(61, 23, 34, a_{15})$
$(12, 40, 76, a_{16})$	$(41, 13, 67, a_{16})$	$(40, 22, 36, a_{17})$	$(31, 73, 77, a_{17})$	$(12, 42, 04, 54)$	$(02, 52, 37, 24)$	$(10, 20, 65, 66)$	$(13, 03, 04, 74)$
$(00, 50, 77, 07)$	$(20, 73, 86, 36)$	$(12, 32, 56, 85)$	$(31, 62, 87, 06)$	$(41, 21, 66, 76)$	$(11, 21, 15, 06)$	$(60, 72, 66, 45)$	$(00, 70, 65, 37)$
$(21, 61, 57, 75)$	$(13, 53, 07, 45)$	$(31, 52, 07, 55)$	$(03, 63, 77, 36)$	$(03, 23, 45, 06)$			

$n = 55$ :

$(\infty, 42, 18, 49)$	$(\infty, 31, 37, 66)$	$(\infty, 10, 14, 05)$	$(41, 70, 37, a_1)$	$(43, 42, 36, a_1)$	$(31, 81, 56, 08)$	$(60, 41, 87, 57)$	$(40, 53, 89, 79)$
$(50, 73, 75, 06)$	$(21, 81, 74, 68)$	$(32, 03, 68, 67)$	$(50, 53, 59, 36)$	$(30, 52, 38, 17)$	$(01, 10, 45, 09)$	$(22, 70, 59, 55)$	$(40, 32, 15, 45)$
$(03, 12, 54, 38)$	$(40, 81, 88, 47)$	$(20, 30, 36, 86)$	$(23, 33, 86, 24)$	$(71, 62, 47, 49)$	$(61, 13, 58, 47)$	$(30, 10, 04, 68)$	$(23, 83, 15, 39)$
$(40, 61, 74, 28)$	$(51, 62, 16, 15)$	$(52, 22, 45, 86)$	$(11, 31, 25, 59)$	$(60, 53, 24, 78)$	$(51, 53, 49, 06)$	$(60, 33, 89, 14)$	$(21, 82, 64, 85)$
$(32, 22, 36, 24)$	$(62, 23, 89, 26)$	$(32, 63, 78, 47)$	$(52, 63, 68, 49)$	$(11, 01, 34, 35)$	$(60, 43, 56, 07)$	$(10, 63, 57, 25)$	$(51, 63, 35, 87)$
$(23, 03, 44, 34)$	$(60, 71, 44, 74)$	$(80, 82, 57, 45)$	$(50, 81, 69, 76)$	$(80, 23, 18, 35)$	$(31, 13, 46, 68)$	$(52, 33, 18, 47)$	$(81, 22, 85, 18)$
$(62, 61, 29, 87)$	$(51, 72, 34, 89)$	$(40, 01, 19, 76)$	$(23, 73, 55, 27)$	$(22, 42, 89, 54)$	$(22, 72, 44, 06)$	$(00, 62, 17, 68)$	

$n = 57$ :

$(\infty, 20, 25, 24)$	$(\infty, 31, 57, 66)$	$(\infty, 22, 39, 08)$	$(52, 53, 36, a_1)$	$(61, 60, 74, a_1)$	$(01, 10, 44, a_2)$	$(73, 62, 06, a_2)$	$(82, 33, 25, a_3)$
$(21, 60, 16, a_3)$	$(20, 83, 45, 48)$	$(50, 53, 58, 46)$	$(50, 72, 48, 87)$	$(32, 30, 88, 86)$	$(43, 63, 74, 78)$	$(20, 61, 58, 64)$	$(50, 31, 89, 27)$
$(12, 22, 15, 79)$	$(30, 22, 47, 77)$	$(31, 61, 77, 68)$	$(42, 00, 16, 79)$	$(20, 33, 47, 89)$	$(01, 41, 58, 16)$	$(30, 53, 75, 87)$	$(40, 73, 76, 84)$
$(51, 63, 75, 89)$	$(62, 53, 74, 14)$	$(12, 62, 19, 64)$	$(20, 60, 69, 46)$	$(11, 81, 86, 89)$	$(13, 73, 46, 59)$	$(52, 13, 56, 07)$	$(41, 32, 05, 59)$
$(20, 72, 88, 17)$	$(61, 02, 67, 89)$	$(40, 33, 05, 28)$	$(13, 63, 79, 67)$	$(41, 33, 74, 18)$	$(20, 80, 55, 86)$	$(20, 40, 35, 39)$	$(11, 63, 64, 45)$
$(01, 62, 57, 49)$	$(71, 53, 08, 85)$	$(41, 42, 34, 66)$	$(71, 13, 55, 49)$	$(20, 10, 04, 74)$	$(61, 22, 48, 16)$	$(40, 11, 47, 58)$	$(82, 63, 65, 38)$
$(31, 42, 05, 14)$	$(71, 40, 69, 25)$	$(42, 01, 24, 64)$	$(22, 82, 36, 28)$	$(23, 52, 84, 48)$	$(32, 63, 37, 17)$	$(31, 12, 35, 25)$	$(31, 33, 86, 17)$
$(00, 53, 77, 59)$							

$n = 59$ :

$(\infty, 71, 07, 26)$	$(\infty, 03, 69, 48)$	$(\infty, 70, 75, 14)$	$(51, 20, 15, a_1)$	$(83, 82, 16, a_1)$	$(01, 40, 64, a_2)$	$(13, 52, 66, a_2)$	$(42, 63, 45, a_3)$
$(30, 41, 76, a_3)$	$(43, 52, 15, a_4)$	$(61, 40, 46, a_4)$	$(70, 61, 64, a_5)$	$(83, 42, 27, a_5)$	$(31, 52, 75, 77)$	$(03, 63, 89, 17)$	$(83, 71, 04, 85)$
$(33, 73, 76, 68)$	$(50, 63, 79, 76)$	$(80, 62, 18, 77)$	$(40, 00, 48, 76)$	$(40, 72, 25, 38)$	$(01, 03, 08, 57)$	$(50, 40, 28, 17)$	$(83, 80, 05, 89)$
$(30, 63, 85, 88)$	$(20, 72, 69, 65)$	$(72, 11, 24, 08)$	$(23, 13, 44, 84)$	$(11, 21, 49, 26)$	$(61, 31, 47, 18)$	$(50, 33, 16, 88)$	$(11, 02, 36, 84)$
$(12, 72, 56, 88)$	$(21, 61, 14, 79)$	$(40, 33, 57, 19)$	$(50, 03, 68, 54)$	$(31, 11, 06, 19)$	$(72, 82, 59, 14)$	$(22, 53, 04, 27)$	$(62, 33, 17, 66)$
$(12, 83, 77, 49)$	$(21, 32, 38, 45)$	$(43, 61, 05, 28)$	$(82, 42, 78, 54)$	$(10, 71, 47, 75)$	$(60, 00, 14, 47)$	$(62, 20, 36, 45)$	$(23, 01, 65, 38)$
$(21, 62, 69, 27)$	$(23, 20, 14, 19)$	$(63, 71, 55, 36)$	$(00, 62, 89, 64)$	$(80, 02, 56, 49)$	$(01, 53, 54, 85)$	$(51, 52, 47, 28)$	$(41, 83, 39, 74)$
$(73, 41, 19, 77)$	$(22, 20, 16, 55)$	$(00, 20, 27, 39)$					

$n = 61$ :

$(\infty, 83, 09, 48)$	$(\infty, 50, 24, 65)$	$(\infty, 61, 17, 26)$	$(41, 30, 44, a_1)$	$(62, 13, 56, a_1)$	$(61, 70, 87, a_2)$	$(42, 53, 45, a_2)$	$(00, 42, 76, a_3)$
$(11, 63, 74, a_3)$	$(42, 60, 04, a_4)$	$(21, 33, 26, a_4)$	$(10, 81, 64, a_5)$	$(63, 12, 05, a_5)$	$(62, 33, 45, a_6)$	$(00, 51, 36, a_6)$	$(51, 43, 76, a_7)$
$(60, 22, 87, a_7)$	$(30, 70, 09, 88)$	$(71, 70, 64, 26)$	$(11, 22, 38, 07)$	$(62, 40, 44, 15)$	$(03, 02, 08, 16)$	$(31, 73, 29, 09)$	$(31, 51, 88, 26)$
$(62, 03, 77, 07)$	$(22, 61, 76, 09)$	$(71, 52, 15, 48)$	$(01, 32, 24, 36)$	$(80, 02, 34, 64)$	$(21, 70, 29, 37)$	$(53, 83, 15, 89)$	$(20, 81, 44, 68)$
$(82, 61, 55, 29)$	$(63, 01, 44, 34)$	$(50, 43, 88, 87)$	$(43, 21, 57, 34)$	$(00, 53, 47, 87)$	$(63, 82, 04, 37)$	$(02, 42, 28, 19)$	$(00, 23, 09, 85)$
$(30, 10, 06, 17)$	$(63, 13, 64, 08)$	$(71, 13, 78, 37)$	$(01, 51, 75, 15)$	$(53, 51, 55, 05)$	$(01, 11, 77, 29)$	$(43, 63, 84, 39)$	$(43, 00, 68, 26)$
$(32, 12, 45, 34)$	$(70, 80, 78, 25)$	$(10, 70, 35, 89)$	$(01, 31, 48, 79)$	$(43, 10, 09, 16)$	$(12, 72, 07, 39)$	$(10, 23, 77, 38)$	$(12, 61, 35, 67)$
$(10, 13, 88, 85)$	$(72, 80, 29, 46)$	$(20, 82, 79, 36)$	$(02, 82, 38, 58)$	$(02, 23, 26, 76)$			

$n = 63$ :

$(\infty, 83, 67, 86)$	$(\infty, 10, 54, 55)$	$(\infty, 72, 29, 58)$	$(21, 73, 34, a_1)$	$(50, 02, 07, a_1)$	$(52, 33, 67, a_2)$	$(31, 70, 34, a_2)$	$(20, 21, 45, a_3)$
$(03, 72, 26, a_3)$	$(22, 83, 26, a_4)$	$(30, 71, 24, a_4)$	$(80, 82, 76, a_5)$	$(63, 61, 34, a_5)$	$(10, 03, 76, a_6)$	$(81, 62, 57, a_6)$	$(03, 51, 17, a_7)$
$(20, 02, 26, a_7)$	$(53, 52, 85, a_8)$	$(00, 21, 04, a_8)$	$(20, 51, 85, a_9)$	$(62, 03, 24, a_9)$	$(13, 63, 28, 75)$	$(71, 82, 27, 38)$	$(21, 71, 16, 46)$
$(80, 42, 25, 38)$	$(60, 00, 16, 59)$	$(50, 23, 76, 24)$	$(41, 42, 34, 67)$	$(72, 51, 65, 84)$	$(10, 20, 08, 37)$	$(11, 23, 28, 27)$	$(60, 40, 79, 26)$
$(63, 01, 54, 45)$	$(51, 42, 39, 74)$	$(01, 42, 55, 05)$	$(12, 80, 77, 57)$	$(42, 50, 84, 09)$	$(51, 80, 19, 45)$	$(70, 42, 65, 85)$	$(42, 02, 79, 16)$
$(21, 31, 05, 58)$	$(51, 81, 38, 66)$	$(10, 53, 47, 34)$	$(21, 13, 26, 19)$	$(33, 30, 44, 68)$	$(21, 82, 89, 66)$	$(03, 71, 19, 57)$	$(43, 10, 79, 84)$
$(70, 83, 75, 78)$	$(72, 12, 48, 75)$	$(03, 13, 78, 89)$	$(12, 23, 68, 66)$	$(21, 72, 88, 57)$	$(60, 20, 78, 27)$	$(41, 23, 69, 48)$	$(03, 52, 86, 29)$
$(50, 31, 86, 28)$	$(42, 22, 48, 44)$	$(40, 63, 29, 25)$	$(43, 11, 17, 19)$	$(02, 82, 19, 64)$	$(70, 61, 57, 79)$	$(03, 23, 25, 47)$	

$n = 65$ :

$(\infty, 03, 89, 36)$	$(\infty, 42, 64, 87)$	$(\infty, 01, 35, 58)$	$(63, 62, 77, a_1)$	$(80, 01, 54, a_1)$	$(12, 60, 07, a_2)$	$(11, 23, 04, a_2)$	$(21, 72, 56, a_3)$
$(50, 63, 84, a_3)$	$(73, 71, 06, a_4)$	$(60, 72, 84, a_4)$	$(63, 52, 76, a_5)$	$(40, 21, 84, a_5)$	$(21, 30, 34, a_6)$	$(72, 23, 75, a_6)$	$(42, 33, 74, a_7)$
$(21, 70, 36, a_7)$	$(72, 51, 37, a_8)$	$(70, 33, 45, a_8)$	$(23, 01, 45, a_9)$	$(00, 52, 26, a_9)$	$(82, 01, 05, a_{10})$	$(03, 50, 57, a_{10})$	$(30, 22, 66, a_{11})$
$(11, 63, 17, a_{11})$	$(21, 02, 29, 85)$	$(20, 02, 25, 78)$	$(32, 02, 37, 36)$	$(01, 81, 08, 86)$	$(41, 21, 69, 87)$	$(42, 01, 59, 19)$	$(00, 32, 48, 74)$
$(40, 61, 18, 57)$	$(50, 70, 09, 05)$	$(61, 03, 14, 87)$	$(60, 83, 87, 09)$	$(12, 01, 66, 89)$	$(23, 13, 47, 58)$	$(42, 23, 78, 38)$	$(73, 22, 74, 29)$
$(10, 83, 76, 29)$	$(40, 11, 25, 05)$	$(11, 72, 78, 47)$	$(13, 70, 56, 69)$	$(02, 72, 86, 55)$	$(61, 13, 77, 85)$	$(00, 31, 16, 09)$	$(22, 32, 79, 14)$
$(83, 13, 86, 89)$	$(60, 82, 29, 55)$	$(70, 73, 64, 49)$	$(61, 60, 27, 49)$	$(70, 20, 34, 66)$	$(70, 42, 08, 85)$	$(32, 72, 57, 08)$	$(40, 50, 37, 58)$
$(13, 63, 05, 88)$	$(00, 83, 78, 35)$	$(80, 41, 86, 78)$	$(20, 83, 67, 58)$	$(83, 22, 55, 19)$	$(13, 21, 76, 18)$	$(71, 31, 58, 54)$	$(41, 23, 25, 74)$
$(02, 33, 04, 64)$							

$n = 67$ :

$(\infty, 12, 69, 08)$	$(\infty, 71, 36, 47)$	$(\infty, 23, 14, 05)$	$(83, 40, 56, a_1)$	$(22, 61, 77, a_1)$	$(13, 70, 46, a_2)$	$(82, 11, 44, a_2)$	$(52, 63, 04, a_3)$
$(30, 71, 77, a_3)$	$(11, 72, 55, a_4)$	$(83, 80, 47, a_4)$	$(80, 21, 57, a_5)$	$(73, 32, 65, a_5)$	$(73, 61, 85, a_6)$	$(50, 12, 44, a_6)$	$(33, 20, 54, a_7)$
$(12, 21, 55, a_7)$	$(00, 81, 74, a_8)$	$(52, 83, 45, a_8)$	$(23, 50, 24, a_9)$	$(11, 12, 07, a_9)$	$(13, 41, 86, a_{10})$	$(52, 80, 07, a_{10})$	$(30, 42, 17, a_{11})$
$(73, 21, 75, a_{11})$	$(61, 83, 17, a_{12})$	$(70, 12, 75, a_{12})$	$(31, 13, 87, a_{13})$	$(82, 10, 86, a_{13})$	$(30, 70, 74, 85)$	$(33, 73, 36, 79)$	$(41, 01, 36, 28)$
$(12, 01, 66, 74)$	$(42, 00, 86, 29)$	$(43, 01, 58, 04)$	$(12, 22, 06, 38)$	$(30, 81, 29, 65)$	$(13, 11, 29, 24)$	$(20, 12, 58, 76)$	$(80, 51, 88, 55)$
$(12, 13, 09, 36)$	$(60, 71, 09, 57)$	$(10, 12, 24, 19)$	$(13, 52, 77, 38)$	$(41, 31, 04, 48)$	$(53, 70, 06, 58)$	$(62, 22, 25, 07)$	$(00, 21, 46, 88)$
$(23, 60, 88, 85)$	$(20, 30, 88, 15)$	$(22, 03, 35, 39)$	$(23, 43, 65, 78)$	$(30, 60, 49, 66)$	$(33, 62, 77, 28)$	$(30, 53, 67, 79)$	$(43, 73, 28, 24)$
$(62, 40, 67, 09)$	$(61, 01, 76, 69)$	$(40, 21, 85, 47)$	$(62, 83, 06, 76)$	$(01, 22, 88, 49)$	$(30, 31, 78, 54)$	$(61, 41, 55, 29)$	$(63, 73, 67, 39)$
$(53, 60, 39, 24)$	$(32, 52, 34, 09)$	$(02, 32, 24, 38)$					

$n = 69$ :

$\langle \infty, 02, 58, 34 \rangle$	$\langle \infty, 70, 17, 55 \rangle$	$\langle \infty, 23, 59, 56 \rangle$	$\langle 52, 01, 55, a_1 \rangle$	$\langle 10, 33, 87, a_1 \rangle$	$\langle 52, 80, 66, a_2 \rangle$	$\langle 43, 81, 04, a_2 \rangle$	$\langle 62, 51, 27, a_3 \rangle$
$\langle 03, 30, 64, a_3 \rangle$	$\langle 32, 00, 55, a_4 \rangle$	$\langle 43, 01, 16, a_4 \rangle$	$\langle 82, 60, 57, a_5 \rangle$	$\langle 53, 81, 46, a_5 \rangle$	$\langle 30, 31, 66, a_6 \rangle$	$\langle 83, 42, 57, a_6 \rangle$	$\langle 03, 02, 04, a_7 \rangle$
$\langle 01, 40, 87, a_7 \rangle$	$\langle 01, 60, 75, a_8 \rangle$	$\langle 82, 73, 14, a_8 \rangle$	$\langle 50, 33, 85, a_9 \rangle$	$\langle 11, 82, 86, a_9 \rangle$	$\langle 53, 51, 55, a_{10} \rangle$	$\langle 52, 60, 14, a_{10} \rangle$	$\langle 81, 23, 36, a_{11} \rangle$
$\langle 00, 42, 14, a_{11} \rangle$	$\langle 11, 32, 25, a_{12} \rangle$	$\langle 40, 33, 56, a_{12} \rangle$	$\langle 81, 20, 86, a_{13} \rangle$	$\langle 32, 43, 65, a_{13} \rangle$	$\langle 30, 82, 26, a_{14} \rangle$	$\langle 03, 21, 44, a_{14} \rangle$	$\langle 13, 00, 57, a_{15} \rangle$
$\langle 61, 12, 24, a_{15} \rangle$	$\langle 22, 53, 67, 48 \rangle$	$\langle 71, 03, 06, 77 \rangle$	$\langle 02, 11, 27, 39 \rangle$	$\langle 43, 80, 64, 28 \rangle$	$\langle 32, 82, 05, 09 \rangle$	$\langle 50, 31, 25, 89 \rangle$	$\langle 00, 41, 19, 84 \rangle$
$\langle 70, 13, 07, 79 \rangle$	$\langle 61, 53, 78, 08 \rangle$	$\langle 11, 61, 87, 49 \rangle$	$\langle 01, 02, 66, 84 \rangle$	$\langle 30, 41, 75, 18 \rangle$	$\langle 30, 12, 86, 78 \rangle$	$\langle 50, 03, 74, 68 \rangle$	$\langle 50, 40, 48, 57 \rangle$
$\langle 60, 80, 85, 86 \rangle$	$\langle 11, 31, 88, 75 \rangle$	$\langle 00, 02, 59, 46 \rangle$	$\langle 02, 63, 49, 78 \rangle$	$\langle 03, 13, 45, 56 \rangle$	$\langle 80, 30, 59, 58 \rangle$	$\langle 63, 13, 69, 59 \rangle$	$\langle 60, 51, 24, 59 \rangle$
$\langle 51, 22, 17, 49 \rangle$	$\langle 83, 32, 19, 17 \rangle$	$\langle 53, 72, 18, 69 \rangle$	$\langle 21, 51, 54, 69 \rangle$	$\langle 12, 02, 55, 18 \rangle$	$\langle 31, 21, 28, 67 \rangle$	$\langle 40, 70, 44, 29 \rangle$	$\langle 23, 43, 34, 15 \rangle$
$\langle 31, 43, 26, 78 \rangle$	$\langle 30, 42, 88, 25 \rangle$	$\langle 03, 63, 75, 07 \rangle$	$\langle 02, 32, 86, 37 \rangle$	$\langle 01, 32, 74, 28 \rangle$			

$n = 71$ :

$\langle \infty, 71, 09, 18 \rangle$	$\langle \infty, 13, 65, 64 \rangle$	$\langle \infty, 20, 47, 26 \rangle$	$\langle 43, 31, 75, a_1 \rangle$	$\langle 60, 62, 06, a_1 \rangle$	$\langle 30, 61, 86, a_2 \rangle$	$\langle 22, 43, 47, a_2 \rangle$	$\langle 00, 61, 76, a_3 \rangle$
$\langle 13, 12, 15, a_3 \rangle$	$\langle 62, 43, 36, a_4 \rangle$	$\langle 51, 00, 34, a_4 \rangle$	$\langle 71, 82, 36, a_5 \rangle$	$\langle 23, 70, 45, a_5 \rangle$	$\langle 70, 13, 66, a_6 \rangle$	$\langle 42, 51, 84, a_6 \rangle$	$\langle 40, 62, 85, a_7 \rangle$
$\langle 21, 43, 84, a_7 \rangle$	$\langle 43, 12, 25, a_8 \rangle$	$\langle 10, 51, 36, a_8 \rangle$	$\langle 21, 30, 37, a_9 \rangle$	$\langle 33, 62, 56, a_9 \rangle$	$\langle 53, 11, 25, a_{10} \rangle$	$\langle 62, 50, 04, a_{10} \rangle$	$\langle 52, 00, 35, a_{11} \rangle$
$\langle 33, 31, 06, a_{11} \rangle$	$\langle 41, 73, 17, a_{12} \rangle$	$\langle 10, 72, 64, a_{12} \rangle$	$\langle 41, 12, 86, a_{13} \rangle$	$\langle 53, 70, 67, a_{13} \rangle$	$\langle 00, 01, 05, a_{14} \rangle$	$\langle 33, 22, 07, a_{14} \rangle$	$\langle 50, 43, 34, a_{15} \rangle$
$\langle 01, 72, 77, a_{15} \rangle$	$\langle 71, 33, 16, a_{16} \rangle$	$\langle 52, 70, 05, a_{16} \rangle$	$\langle 63, 01, 57, a_{17} \rangle$	$\langle 10, 52, 34, a_{17} \rangle$	$\langle 01, 51, 87, 48 \rangle$	$\langle 10, 23, 79, 29 \rangle$	$\langle 00, 11, 85, 57 \rangle$
$\langle 12, 52, 47, 68 \rangle$	$\langle 10, 73, 89, 65 \rangle$	$\langle 51, 02, 57, 04 \rangle$	$\langle 12, 51, 69, 77 \rangle$	$\langle 42, 03, 88, 46 \rangle$	$\langle 01, 73, 59, 14 \rangle$	$\langle 10, 50, 68, 87 \rangle$	$\langle 11, 32, 59, 28 \rangle$
$\langle 30, 11, 89, 06 \rangle$	$\langle 61, 01, 09, 84 \rangle$	$\langle 63, 03, 06, 15 \rangle$	$\langle 20, 50, 67, 35 \rangle$	$\langle 61, 53, 24, 66 \rangle$	$\langle 00, 20, 14, 68 \rangle$	$\langle 41, 31, 65, 08 \rangle$	$\langle 03, 23, 49, 28 \rangle$
$\langle 32, 73, 49, 19 \rangle$	$\langle 20, 30, 29, 18 \rangle$	$\langle 51, 30, 38, 58 \rangle$	$\langle 02, 12, 24, 64 \rangle$	$\langle 83, 43, 64, 67 \rangle$	$\langle 70, 62, 86, 19 \rangle$	$\langle 81, 61, 55, 59 \rangle$	$\langle 00, 03, 38, 04 \rangle$
$\langle 52, 51, 78, 15 \rangle$	$\langle 10, 63, 74, 59 \rangle$	$\langle 22, 42, 28, 15 \rangle$	$\langle 23, 32, 68, 08 \rangle$	$\langle 12, 72, 27, 79 \rangle$	$\langle 10, 42, 39, 56 \rangle$	$\langle 03, 13, 57, 68 \rangle$	

$n = 73$ :

$\langle \infty, 13, 55, 66 \rangle$	$\langle \infty, 41, 110, 59 \rangle$	$\langle \infty, 30, 67, 011 \rangle$	$\langle \infty, 02, 18, 14 \rangle$	$\langle 31, 50, 86, a_1 \rangle$	$\langle 42, 73, 87, a_1 \rangle$	$\langle 21, 80, 25, 19 \rangle$	$\langle 42, 23, 89, 210 \rangle$
$\langle 01, 53, 74, 85 \rangle$	$\langle 12, 20, 34, 06 \rangle$	$\langle 31, 33, 38, 510 \rangle$	$\langle 13, 70, 08, 67 \rangle$	$\langle 10, 03, 111, 85 \rangle$	$\langle 10, 33, 06, 77 \rangle$	$\langle 42, 02, 48, 04 \rangle$	$\langle 60, 31, 59, 68 \rangle$
$\langle 13, 83, 38, 36 \rangle$	$\langle 11, 23, 58, 59 \rangle$	$\langle 70, 31, 810, 77 \rangle$	$\langle 10, 82, 59, 78 \rangle$	$\langle 21, 02, 17, 89 \rangle$	$\langle 31, 23, 710, 64 \rangle$	$\langle 32, 33, 410, 17 \rangle$	$\langle 00, 52, 38, 15 \rangle$
$\langle 00, 11, 410, 74 \rangle$	$\langle 32, 02, 39, 46 \rangle$	$\langle 50, 52, 44, 19 \rangle$	$\langle 11, 10, 88, 76 \rangle$	$\langle 13, 03, 14, 86 \rangle$	$\langle 52, 23, 69, 17 \rangle$	$\langle 20, 80, 410, 46 \rangle$	$\langle 22, 73, 88, 48 \rangle$
$\langle 12, 40, 110, 310 \rangle$	$\langle 22, 70, 35, 111 \rangle$	$\langle 52, 81, 35, 47 \rangle$	$\langle 22, 12, 55, 811 \rangle$	$\langle 23, 41, 711, 04 \rangle$	$\langle 20, 33, 18, 84 \rangle$	$\langle 32, 11, 19, 06 \rangle$	$\langle 11, 62, 28, 86 \rangle$
$\langle 40, 72, 56, 211 \rangle$	$\langle 31, 22, 311, 511 \rangle$	$\langle 43, 21, 711, 46 \rangle$	$\langle 13, 53, 65, 311 \rangle$	$\langle 60, 33, 15, 211 \rangle$	$\langle 53, 21, 011, 44 \rangle$	$\langle 42, 63, 76, 611 \rangle$	$\langle 31, 42, 46, 011 \rangle$
$\langle 11, 81, 45, 04 \rangle$	$\langle 72, 71, 45, 77 \rangle$	$\langle 32, 20, 67, 68 \rangle$	$\langle 20, 00, 44, 411 \rangle$	$\langle 10, 60, 16, 410 \rangle$	$\langle 00, 22, 04, 54 \rangle$	$\langle 22, 63, 87, 510 \rangle$	$\langle 11, 42, 69, 14 \rangle$
$\langle 10, 13, 35, 68 \rangle$	$\langle 10, 31, 110, 211 \rangle$	$\langle 21, 61, 45, 09 \rangle$	$\langle 01, 43, 15, 111 \rangle$	$\langle 00, 43, 69, 77 \rangle$	$\langle 42, 22, 810, 45 \rangle$	$\langle 30, 13, 69, 19 \rangle$	$\langle 42, 01, 06, 411 \rangle$
$\langle 40, 31, 57, 74 \rangle$	$\langle 41, 13, 510, 410 \rangle$	$\langle 40, 50, 45, 59 \rangle$	$\langle 23, 53, 39, 27 \rangle$	$\langle 51, 21, 48, 87 \rangle$	$\langle 12, 03, 05, 610 \rangle$	$\langle 02, 13, 810, 44 \rangle$	$\langle 10, 51, 67, 37 \rangle$
$\langle 01, 11, 46, 68 \rangle$							

**B. SFSs of type  $(4, 2)^a(2, 4)^b$  with  $a + b \in \{5, 6, 7, 8, 9\}$**

To save space, we only list the non-empty cells of the SFSs. We use  $(a, b; i, j)$  to denote a cell which is indexed by  $(i, j)$  and contains a pair  $\{a, b\}$ . We use  $I_n$  to denote the set  $\{0, 1, 2, \dots, n-1\}$ .

**Lemma 8.8:** There exists an SFS of type  $(4, 2)^a(2, 2)^{5-a}$  for each  $a \in \{0, 1, 2, 3, 4, 5\}$ .

**Proof:** Let  $V = I_{10}$  and  $S = I_{10+2a}$ .  $V$  can be partitioned as  $V = \cup_{i=0}^4 \{2i, 2i+1\}$  and  $S$  can be partitioned as  $S = (\cup_{i=0}^{a-1} \{4i, 4i+1, 4i+2, 4i+3\}) \cup (\cup_{i=a}^4 \{2i, 2i+1\})$ . The required SFSs are presented as follows.

$a = 0$ :

$(1, 6; 9, 3)$	$(1, 8; 7, 2)$	$(5, 3; 9, 6)$	$(5, 9; 3, 0)$	$(0, 8; 3, 5)$	$(2, 4; 0, 7)$	$(6, 9; 4, 2)$	$(9, 3; 1, 7)$	$(2, 7; 9, 5)$	$(6, 3; 0, 5)$
$(0, 4; 2, 9)$	$(5, 0; 8, 7)$	$(2, 6; 8, 1)$	$(0, 2; 4, 6)$	$(7, 8; 4, 0)$	$(9, 1; 6, 5)$	$(7, 4; 3, 8)$	$(1, 3; 4, 8)$	$(4, 8; 1, 6)$	$(5, 7; 1, 2)$

$a = 1$ :

$(7, 0; 7, 11)$	$(6, 3; 10, 1)$	$(0, 4; 8, 10)$	$(6, 9; 6, 3)$	$(4, 2; 1, 11)$	$(0, 8; 5, 6)$	$(6, 1; 7, 4)$	$(8, 2; 8, 3)$	$(3, 8; 7, 0)$
$(8, 5; 4, 1)$	$(4, 8; 2, 9)$	$(6, 4; 5, 0)$	$(7, 9; 1, 5)$	$(4, 7; 4, 3)$	$(5, 3; 9, 3)$	$(0, 9; 4, 9)$	$(9, 5; 8, 0)$	$(5, 1; 10, 5)$
$(1, 3; 11, 8)$	$(9, 2; 7, 2)$	$(7, 2; 10, 0)$	$(1, 2; 6, 9)$	$(3, 7; 2, 6)$	$(5, 6; 2, 11)$			

$a = 2$ :

$(2, 5; 2, 10)$	$(6, 9; 4, 2)$	$(8, 7; 6, 2)$	$(4, 2; 0, 13)$	$(6, 3; 9, 3)$	$(8, 2; 11, 9)$	$(5, 6; 6, 0)$	$(7, 1; 8, 4)$	$(0, 7; 7, 9)$
$(2, 6; 1, 12)$	$(0, 9; 6, 11)$	$(8, 6; 8, 7)$	$(9, 7; 5, 0)$	$(9, 1; 9, 10)$	$(0, 4; 5, 10)$	$(0, 5; 4, 13)$	$(9, 2; 8, 3)$	$(4, 8; 4, 3)$
$(4, 1; 6, 12)$	$(3, 7; 13, 1)$	$(5, 7; 12, 3)$	$(8, 3; 10, 0)$	$(3, 4; 2, 11)$	$(3, 0; 8, 12)$	$(4, 9; 1, 7)$	$(1, 5; 7, 11)$	$(1, 6; 5, 13)$
$(5, 8; 1, 5)$								

$a = 3$ :

$(4, 0; 7, 15)$	$(9, 3; 3, 10)$	$(3, 7; 15, 11)$	$(7, 8; 7, 3)$	$(9, 0; 8, 6)$	$(5, 2; 15, 3)$	$(7, 5; 14, 6)$	$(8, 3; 12, 9)$	$(4, 3; 14, 0)$
$(1, 7; 8, 4)$	$(8, 2; 0, 8)$	$(6, 3; 8, 1)$	$(6, 4; 4, 3)$	$(6, 1; 9, 15)$	$(5, 6; 7, 0)$	$(5, 9; 4, 1)$	$(4, 1; 6, 12)$	$(7, 2; 1, 10)$
$(2, 9; 2, 12)$	$(4, 7; 5, 2)$	$(0, 8; 4, 11)$	$(7, 9; 0, 9)$	$(8, 6; 2, 6)$	$(6, 0; 14, 10)$	$(5, 3; 13, 2)$	$(9, 6; 5, 11)$	$(2, 1; 11, 14)$
$(0, 5; 12, 5)$	$(8, 4; 1, 13)$	$(0, 2; 9, 13)$	$(1, 8; 5, 10)$	$(1, 9; 7, 13)$				

$a = 4$ :

$(4, 0; 13, 6)$	$(6, 1; 7, 11)$	$(7, 5; 6, 3)$	$(9, 2; 13, 10)$	$(8, 2; 11, 3)$	$(2, 1; 14, 17)$	$(4, 7; 16, 7)$	$(9, 1; 9, 6)$	$(8, 6; 6, 1)$
$(3, 1; 13, 8)$	$(6, 9; 5, 0)$	$(8, 0; 7, 14)$	$(3, 6; 10, 2)$	$(0, 9; 15, 8)$	$(4, 1; 5, 15)$	$(1, 8; 10, 12)$	$(2, 4; 0, 12)$	$(5, 3; 0, 14)$
$(9, 4; 1, 14)$	$(5, 2; 1, 15)$	$(7, 0; 5, 10)$	$(8, 7; 0, 8)$	$(5, 1; 4, 16)$	$(6, 2; 8, 16)$	$(3, 0; 16, 11)$	$(9, 7; 11, 4)$	$(0, 5; 12, 17)$
$(6, 0; 9, 4)$	$(8, 5; 5, 13)$	$(9, 5; 2, 7)$	$(6, 4; 17, 3)$	$(9, 3; 12, 3)$	$(4, 8; 2, 4)$	$(7, 3; 1, 17)$	$(2, 7; 2, 9)$	$(3, 8; 9, 15)$

$a = 5$ :

$(0, 7; 17, 7)$	$(4, 3; 1, 17)$	$(3, 5; 3, 12)$	$(5, 8; 0, 14)$	$(8, 7; 4, 2)$	$(5, 7; 5, 16)$	$(0, 3; 9, 16)$	$(2, 6; 17, 11)$	$(8, 4; 3, 7)$
$(0, 4; 19, 13)$	$(3, 8; 10, 15)$	$(1, 6; 6, 18)$	$(1, 2; 15, 16)$	$(1, 4; 14, 5)$	$(6, 0; 4, 8)$	$(5, 0; 15, 18)$	$(6, 4; 2, 16)$	$(6, 5; 1, 7)$
$(6, 9; 10, 3)$	$(9, 2; 0, 8)$	$(1, 3; 11, 13)$	$(9, 3; 14, 2)$	$(8, 6; 5, 9)$	$(7, 1; 10, 19)$	$(2, 5; 19, 2)$	$(0, 9; 12, 5)$	$(2, 8; 13, 1)$
$(2, 7; 9, 3)$	$(0, 8; 6, 11)$	$(5, 9; 13, 6)$	$(9, 1; 9, 7)$	$(9, 4; 15, 4)$	$(2, 0; 14, 10)$	$(4, 2; 18, 12)$	$(7, 9; 11, 1)$	$(1, 8; 12, 8)$
$(5, 1; 4, 17)$	$(3, 7; 8, 18)$	$(3, 6; 0, 19)$	$(4, 7; 0, 6)$					

**Lemma 8.9:** There exists an SFS of type  $(4, 2)^a(2, 2)^{6-a}$  for each  $a \in \{0, 1, \dots, 6\}$ .



*Proof:* Let  $V = I_{12}$  and  $S = I_{12+2a}$ .  $V$  can be partitioned as  $V = \cup_{i=0}^5 \{2i, 2i+1\}$  and  $S$  can be partitioned as  $S = (\cup_{i=0}^{a-1} \{4i, 4i+1, 4i+2, 4i+3\}) \cup (\cup_{i=a}^5 \{2i, 2i+1\})$ . The required SFSs are presented as follows.

$a = 0$ :

(4, 7; 10, 1)	(5, 0; 11, 7)	(3, 1; 5, 11)	(8, 4; 11, 6)	(9, 5; 3, 10)	(6, 3; 10, 0)	(10, 8; 0, 5)	(11, 9; 6, 5)	(3, 0; 6, 9)
(10, 7; 3, 4)	(2, 1; 7, 10)	(11, 5; 9, 1)	(7, 2; 5, 9)	(0, 8; 10, 4)	(5, 7; 0, 8)	(1, 5; 2, 6)	(0, 6; 5, 3)	(6, 8; 1, 2)
(2, 9; 4, 0)	(3, 11; 8, 4)	(2, 10; 6, 1)	(3, 9; 1, 7)	(9, 7; 2, 11)	(1, 6; 4, 9)	(4, 1; 8, 3)	(6, 2; 11, 8)	(8, 11; 3, 7)
(4, 10; 9, 7)	(0, 10; 2, 8)	(4, 11; 0, 2)						

$a = 1$ :

(1, 10; 5, 6)	(1, 7; 11, 4)	(0, 6; 11, 7)	(1, 3; 13, 10)	(5, 6; 0, 5)	(11, 3; 9, 7)	(2, 10; 1, 7)	(5, 10; 2, 10)	(7, 4; 13, 1)
(2, 0; 9, 13)	(4, 3; 12, 0)	(0, 8; 12, 4)	(11, 9; 2, 5)	(4, 2; 11, 2)	(7, 9; 12, 3)	(9, 10; 0, 9)	(10, 3; 11, 3)	(4, 0; 5, 10)
(9, 1; 7, 8)	(8, 7; 7, 5)	(8, 4; 9, 3)	(9, 5; 13, 4)	(6, 8; 2, 13)	(2, 5; 8, 3)	(5, 11; 11, 1)	(6, 2; 12, 10)	(7, 3; 2, 6)
(11, 6; 4, 3)	(9, 6; 6, 1)	(8, 3; 8, 1)	(4, 10; 8, 4)	(7, 11; 0, 10)	(0, 11; 6, 8)	(1, 5; 9, 12)	(2, 8; 0, 6)	

$a = 2$ :

(4, 8; 14, 1)	(9, 3; 1, 9)	(7, 4; 5, 12)	(2, 9; 11, 3)	(8, 2; 2, 8)	(11, 8; 6, 0)	(3, 10; 11, 2)	(0, 7; 13, 6)	(1, 8; 9, 5)
(11, 1; 4, 8)	(6, 0; 9, 4)	(4, 11; 7, 11)	(0, 8; 11, 15)	(6, 11; 1, 5)	(3, 11; 12, 3)	(3, 4; 0, 10)	(1, 5; 6, 11)	(2, 11; 10, 13)
(2, 7; 1, 15)	(0, 5; 7, 10)	(8, 6; 3, 7)	(9, 5; 5, 2)	(6, 5; 0, 14)	(3, 5; 15, 13)	(7, 9; 7, 0)	(2, 0; 14, 12)	(8, 10; 4, 10)
(10, 5; 12, 1)	(6, 9; 8, 6)	(2, 10; 0, 9)	(3, 7; 8, 14)	(1, 6; 12, 15)	(4, 9; 15, 4)	(5, 7; 4, 3)	(1, 9; 10, 14)	(7, 11; 9, 2)
(4, 10; 6, 3)	(0, 10; 5, 8)	(1, 10; 7, 13)	(4, 6; 2, 13)					

$a = 3$ :

(6, 10; 10, 4)	(4, 10; 1, 5)	(6, 2; 9, 0)	(9, 6; 16, 1)	(9, 2; 8, 17)	(11, 5; 0, 4)	(8, 11; 11, 12)	(7, 2; 11, 1)	(3, 11; 1, 8)
(9, 0; 12, 9)	(10, 0; 11, 14)	(8, 2; 10, 2)	(1, 3; 16, 12)	(0, 3; 10, 13)	(8, 5; 1, 7)	(11, 2; 13, 3)	(8, 0; 16, 4)	(6, 3; 15, 2)
(1, 10; 15, 8)	(5, 2; 16, 14)	(11, 6; 7, 14)	(8, 1; 6, 9)	(7, 3; 17, 14)	(10, 9; 13, 7)	(5, 10; 12, 2)	(0, 6; 8, 6)	(11, 9; 10, 5)
(6, 1; 11, 5)	(4, 2; 15, 12)	(8, 6; 17, 3)	(9, 3; 11, 0)	(4, 11; 2, 6)	(7, 1; 7, 10)	(4, 0; 7, 17)	(7, 10; 0, 6)	(4, 1; 4, 14)
(4, 8; 0, 13)	(10, 3; 3, 9)	(11, 7; 15, 9)	(8, 7; 8, 5)	(5, 1; 13, 17)	(0, 5; 15, 5)	(4, 7; 3, 16)	(5, 9; 3, 6)	(7, 9; 2, 4)

$a = 4$ :

(8, 11; 2, 12)	(3, 8; 10, 19)	(5, 1; 14, 6)	(4, 11; 14, 5)	(11, 2; 1, 11)	(4, 10; 12, 1)	(10, 7; 7, 11)	(9, 3; 11, 15)	(9, 7; 1, 10)
(0, 8; 18, 11)	(5, 7; 2, 19)	(9, 5; 12, 5)	(2, 7; 3, 8)	(1, 10; 17, 13)	(3, 6; 0, 18)	(0, 2; 16, 15)	(6, 2; 9, 2)	(11, 9; 3, 6)
(4, 7; 4, 18)	(5, 2; 18, 17)	(0, 10; 5, 10)	(10, 5; 0, 16)	(8, 2; 0, 14)	(11, 3; 16, 8)	(6, 0; 19, 7)	(6, 8; 1, 5)	(7, 11; 0, 9)
(1, 6; 11, 4)	(11, 6; 17, 10)	(4, 3; 2, 17)	(1, 7; 16, 5)	(4, 6; 16, 3)	(1, 8; 7, 8)	(9, 0; 14, 8)	(8, 5; 3, 4)	(9, 4; 7, 0)
(2, 1; 12, 10)	(11, 5; 15, 7)	(9, 2; 13, 19)	(10, 8; 15, 9)	(8, 4; 6, 13)	(10, 6; 8, 6)	(4, 1; 15, 19)	(3, 10; 14, 3)	(1, 9; 9, 18)
(9, 10; 2, 4)	(7, 0; 6, 17)	(0, 11; 4, 13)	(0, 3; 9, 12)	(3, 5; 1, 13)				

$a = 5$ :

(5, 9; 1, 13)	(10, 0; 16, 8)	(9, 0; 5, 21)	(4, 10; 1, 7)	(5, 3; 20, 3)	(1, 3; 16, 15)	(6, 2; 21, 1)	(9, 4; 3, 6)	(7, 10; 10, 6)
(2, 9; 8, 0)	(0, 4; 17, 15)	(11, 8; 10, 7)	(1, 8; 13, 6)	(8, 5; 15, 0)	(7, 5; 2, 19)	(10, 6; 18, 2)	(5, 11; 17, 6)	(4, 11; 2, 13)
(6, 3; 17, 10)	(1, 5; 18, 21)	(2, 8; 20, 2)	(4, 1; 14, 20)	(9, 10; 11, 15)	(1, 2; 9, 19)	(2, 0; 14, 10)	(6, 0; 6, 20)	(2, 11; 15, 18)
(11, 0; 19, 11)	(4, 2; 16, 12)	(7, 8; 1, 5)	(9, 11; 4, 14)	(5, 6; 4, 16)	(7, 11; 3, 16)	(1, 9; 10, 12)	(1, 10; 17, 4)	(10, 5; 5, 14)
(4, 6; 5, 19)	(7, 0; 4, 9)	(3, 8; 14, 11)	(3, 10; 0, 19)	(8, 4; 21, 4)	(9, 3; 9, 2)	(8, 6; 8, 3)	(0, 5; 7, 12)	(7, 2; 11, 17)
(6, 11; 0, 9)	(10, 2; 13, 3)	(1, 6; 7, 11)	(9, 7; 20, 7)	(11, 1; 8, 5)	(10, 8; 12, 9)	(3, 7; 8, 21)	(0, 3; 13, 18)	(3, 11; 1, 12)
(4, 7; 0, 18)								

$a = 6$ :

(8, 5; 3, 7)	(10, 2; 17, 0)	(9, 10; 3, 8)	(5, 0; 4, 18)	(0, 8; 12, 11)	(1, 11; 4, 11)	(6, 3; 0, 18)	(7, 9; 1, 9)	(8, 10; 9, 5)
(6, 9; 2, 21)	(4, 7; 18, 5)	(4, 8; 21, 0)	(7, 8; 22, 10)	(1, 6; 8, 17)	(4, 6; 20, 3)	(7, 2; 19, 8)	(7, 11; 0, 16)	(2, 11; 18, 3)
(8, 6; 23, 4)	(3, 7; 3, 11)	(10, 6; 7, 11)	(1, 4; 23, 14)	(6, 5; 6, 1)	(7, 10; 4, 2)	(1, 3; 19, 9)	(8, 11; 6, 8)	(11, 3; 17, 14)
(5, 3; 21, 13)	(4, 2; 13, 2)	(2, 9; 11, 23)	(4, 10; 6, 12)	(1, 7; 21, 14)	(0, 4; 7, 17)	(3, 0; 23, 8)	(8, 1; 13, 20)	(9, 4; 15, 4)
(2, 1; 12, 22)	(10, 1; 18, 15)	(0, 10; 16, 13)	(5, 7; 23, 17)	(9, 0; 5, 17)	(6, 2; 9, 16)	(1, 5; 16, 5)	(8, 2; 1, 14)	(5, 11; 12, 2)
(7, 0; 6, 20)	(5, 9; 0, 22)	(0, 11; 9, 15)	(11, 4; 1, 19)	(4, 3; 22, 16)	(5, 10; 19, 14)	(9, 3; 12, 20)	(5, 2; 20, 15)	(11, 9; 7, 13)
(9, 1; 6, 10)	(10, 3; 10, 1)	(8, 3; 15, 2)	(0, 6; 19, 22)	(0, 2; 10, 21)	(6, 11; 5, 10)			

**Lemma 8.10:** There exists an SFS of type  $(4, 2)^a(2, 2)^{7-a}$  for each  $a \in \{0, 1, \dots, 7\}$ . ■

*Proof:* Let  $V = I_{14}$  and  $S = I_{14+2a}$ .  $V$  can be partitioned as  $V = \cup_{i=0}^6 \{2i, 2i+1\}$  and  $S$  can be partitioned as  $S = (\cup_{i=0}^{a-1} \{4i, 4i+1, 4i+2, 4i+3\}) \cup (\cup_{i=a}^6 \{2i, 2i+1\})$ . The required SFSs are presented as follows.

$a = 0$ :

(3, 7; 12, 0)	(11, 1; 4, 6)	(7, 11; 8, 2)	(13, 2; 9, 0)	(9, 5; 11, 3)	(9, 10; 2, 6)	(3, 6; 11, 9)	(5, 10; 7, 12)	(6, 5; 10, 0)
(8, 3; 6, 10)	(0, 12; 3, 6)	(2, 7; 11, 1)	(12, 8; 0, 11)	(9, 1; 10, 12)	(2, 0; 8, 10)	(0, 3; 7, 13)	(13, 7; 4, 10)	(4, 12; 7, 10)
(3, 9; 1, 4)	(8, 1; 2, 7)	(2, 4; 13, 6)	(13, 0; 2, 11)	(6, 1; 13, 3)	(4, 1; 11, 8)	(4, 10; 3, 0)	(2, 9; 5, 7)	(12, 5; 2, 1)
(13, 5; 8, 6)	(11, 0; 12, 5)	(1, 12; 9, 5)	(4, 6; 2, 12)	(12, 6; 4, 8)	(11, 13; 3, 7)	(3, 10; 8, 5)	(4, 11; 1, 9)	(9, 11; 0, 13)
(8, 7; 3, 5)	(6, 13; 1, 5)	(5, 7; 9, 13)	(10, 0; 4, 9)	(2, 8; 4, 12)	(8, 10; 1, 13)			

$a = 1$ :

(11, 5; 2, 11)	(2, 12; 1, 12)	(11, 6; 6, 1)	(1, 11; 14, 7)	(12, 9; 0, 4)	(10, 5; 14, 10)	(1, 10; 15, 8)	(1, 5; 13, 4)	(11, 13; 4, 8)
(9, 1; 12, 5)	(0, 6; 12, 11)	(8, 4; 13, 3)	(12, 7; 7, 13)	(5, 2; 3, 15)	(2, 9; 6, 14)	(13, 2; 9, 13)	(6, 4; 15, 4)	(11, 8; 5, 15)
(8, 12; 6, 2)	(13, 7; 12, 2)	(5, 8; 9, 12)	(2, 11; 0, 10)	(11, 9; 9, 3)	(0, 12; 10, 5)	(12, 3; 3, 8)	(4, 3; 12, 0)	(9, 5; 8, 1)
(13, 10; 0, 6)	(13, 4; 11, 5)	(0, 9; 13, 15)	(8, 10; 1, 4)	(13, 6; 10, 3)	(6, 9; 7, 2)	(8, 0; 8, 7)	(3, 6; 14, 13)	(0, 4; 14, 9)
(10, 3; 9, 2)	(13, 3; 1, 7)	(4, 2; 2, 8)	(3, 1; 10, 6)	(7, 3; 15, 11)	(7, 4; 10, 1)	(7, 0; 6, 4)	(12, 1; 9, 11)	(2, 10; 7, 11)
(7, 8; 0, 14)	(5, 6; 0, 5)	(7, 10; 3, 5)						

$a = 2$ :

(11, 4; 3, 7)	(7, 12; 12, 3)	(6, 9; 4, 3)	(12, 6; 2, 6)	(8, 10; 6, 1)	(2, 1; 10, 8)	(7, 1; 7, 13)	(2, 7; 17, 15)	(5, 13; 5, 2)
(8, 5; 7, 17)	(0, 12; 4, 11)	(10, 12; 8, 7)	(12, 5; 0, 15)	(0, 6; 7, 14)	(10, 6; 12, 16)	(8, 7; 0, 5)	(8, 2; 11, 3)	(0, 11; 16, 10)
(5, 9; 1, 10)	(9, 13; 7, 15)	(4, 10; 10, 5)	(5, 2; 14, 12)	(2, 10; 9, 2)	(4, 1; 15, 16)	(13, 10; 4, 0)	(2, 6; 13, 1)	(11, 3; 2, 17)
(10, 9; 11, 17)	(2, 9; 0, 16)	(6, 11; 0, 8)	(4, 3; 11, 0)	(8, 13; 10, 14)	(9, 7; 2, 8)	(11, 13; 13, 11)	(11, 12; 1, 5)	(0, 4; 13, 17)
(8, 4; 4, 2)	(5, 7; 16, 4)	(1, 11; 12, 4)	(8, 6; 15, 9)	(7, 3; 1, 14)	(3, 12; 10, 13)	(11, 7; 6, 9)	(3, 0; 15, 12)	(13, 4; 12, 1)
(0, 9; 5, 9)	(6, 1; 5, 17)	(1, 5; 6, 11)	(10, 5; 13, 3)	(1, 12; 9, 14)	(4, 9; 6, 14)	(13, 3; 3, 9)	(0, 13; 6, 8)	(3, 8; 8, 16)



$a = 3$ :

(6, 3; 18, 15)	(4, 3; 16, 12)	(11, 0; 11, 5)	(4, 12; 13, 1)	(11, 2; 2, 18)	(10, 6; 0, 6)	(3, 8; 0, 11)	(5, 8; 5, 18)	(4, 11; 15, 6)
(7, 8; 8, 17)	(12, 8; 2, 16)	(8, 1; 6, 19)	(8, 13; 13, 3)	(12, 11; 4, 8)	(5, 11; 12, 0)	(2, 0; 13, 19)	(4, 0; 18, 14)	(1, 10; 4, 18)
(13, 9; 4, 0)	(7, 2; 3, 9)	(6, 4; 3, 5)	(3, 7; 14, 2)	(12, 0; 6, 9)	(9, 3; 1, 8)	(10, 12; 11, 12)	(10, 3; 13, 10)	(11, 9; 3, 10)
(12, 7; 5, 0)	(7, 2; 18, 16)	(1, 6; 10, 17)	(10, 5; 3, 19)	(0, 9; 12, 17)	(0, 13; 7, 15)	(1, 7; 11, 7)	(1, 5; 15, 16)	(4, 9; 2, 7)
(13, 7; 10, 6)	(5, 9; 6, 13)	(3, 12; 3, 17)	(0, 8; 10, 4)	(10, 8; 7, 9)	(6, 9; 19, 11)	(7, 10; 1, 15)	(10, 2; 8, 14)	(7, 4; 4, 19)
(3, 11; 9, 19)	(13, 1; 8, 12)	(6, 11; 1, 7)	(4, 2; 17, 0)	(13, 6; 9, 14)	(10, 13; 5, 2)	(0, 6; 8, 16)	(12, 2; 15, 10)	(1, 9; 9, 5)
(6, 5; 2, 4)	(5, 12; 14, 7)	(1, 11; 14, 13)	(13, 2; 11, 16)	(2, 8; 1, 12)	(5, 13; 1, 17)			

$a = 4$ :

(2, 5; 1, 18)	(12, 6; 1, 17)	(4, 9; 19, 2)	(10, 3; 10, 0)	(6, 5; 2, 20)	(10, 0; 12, 8)	(1, 7; 11, 20)	(9, 0; 9, 20)	(6, 4; 18, 21)
(2, 1; 9, 14)	(8, 10; 6, 15)	(10, 12; 11, 5)	(0, 5; 15, 19)	(3, 13; 15, 18)	(10, 9; 1, 14)	(2, 9; 21, 11)	(1, 6; 10, 6)	(13, 4; 1, 7)
(13, 2; 17, 8)	(4, 0; 14, 4)	(5, 7; 17, 7)	(5, 3; 21, 16)	(10, 1; 17, 21)	(8, 1; 4, 13)	(8, 6; 0, 9)	(10, 7; 4, 16)	(4, 12; 6, 16)
(0, 8; 7, 18)	(13, 0; 6, 13)	(7, 8; 2, 8)	(11, 6; 4, 8)	(12, 1; 18, 8)	(9, 5; 13, 5)	(0, 11; 11, 17)	(4, 11; 3, 20)	(1, 9; 12, 7)
(4, 2; 0, 12)	(9, 3; 8, 3)	(13, 9; 4, 10)	(4, 1; 15, 5)	(3, 8; 20, 1)	(4, 3; 13, 17)	(9, 11; 0, 15)	(3, 11; 2, 14)	(8, 13; 12, 11)
(5, 13; 0, 14)	(12, 5; 3, 4)	(2, 12; 15, 2)	(7, 13; 3, 5)	(3, 6; 19, 11)	(8, 11; 21, 5)	(11, 5; 6, 12)	(12, 8; 10, 14)	(2, 10; 20, 13)
(3, 12; 9, 12)	(10, 6; 7, 3)	(8, 2; 3, 19)	(6, 0; 5, 16)	(12, 11; 7, 13)	(1, 13; 16, 19)	(7, 9; 6, 18)	(7, 12; 19, 0)	(13, 10; 2, 9)
(11, 2; 10, 16)	(0, 7; 10, 21)	(7, 11; 1, 9)						

$a = 5$ :

(2, 6; 18, 21)	(6, 1; 7, 9)	(10, 13; 9, 17)	(12, 8; 5, 13)	(13, 8; 14, 8)	(3, 4; 22, 17)	(2, 9; 13, 10)	(6, 4; 5, 1)	(2, 8; 23, 2)
(7, 11; 3, 5)	(5, 11; 19, 1)	(4, 0; 19, 6)	(5, 0; 13, 18)	(3, 12; 21, 16)	(12, 0; 15, 8)	(13, 0; 5, 16)	(13, 2; 19, 3)	(1, 13; 6, 12)
(10, 9; 5, 14)	(10, 0; 7, 22)	(0, 11; 9, 14)	(7, 0; 10, 20)	(13, 3; 1, 10)	(12, 7; 18, 7)	(7, 1; 4, 17)	(6, 12; 19, 10)	(13, 5; 7, 15)
(2, 11; 8, 17)	(0, 6; 23, 17)	(5, 12; 17, 20)	(12, 4; 0, 4)	(2, 4; 20, 14)	(8, 7; 9, 6)	(7, 5; 2, 16)	(8, 5; 12, 22)	(3, 1; 20, 19)
(9, 5; 21, 6)	(9, 13; 20, 4)	(5, 10; 3, 4)	(3, 6; 8, 3)	(5, 3; 14, 0)	(10, 8; 1, 11)	(12, 11; 6, 2)	(12, 9; 3, 12)	(4, 1; 13, 21)
(5, 1; 23, 5)	(1, 10; 18, 10)	(8, 0; 21, 4)	(7, 13; 0, 21)	(2, 7; 11, 22)	(1, 9; 8, 22)	(0, 3; 12, 11)	(10, 7; 19, 8)	(4, 10; 23, 15)
(1, 12; 14, 11)	(11, 8; 10, 15)	(6, 9; 11, 2)	(10, 2; 0, 12)	(13, 11; 13, 11)	(8, 6; 0, 20)	(9, 3; 9, 15)	(7, 9; 23, 1)	(4, 8; 7, 3)
(11, 3; 23, 18)	(4, 13; 2, 18)	(9, 11; 7, 0)	(10, 6; 6, 16)	(1, 2; 15, 16)	(2, 12; 1, 9)	(3, 10; 2, 13)	(4, 11; 12, 16)	(6, 11; 4, 22)

$a = 6$ :

(2, 12; 0, 23)	(1, 3; 12, 11)	(10, 9; 14, 2)	(1, 5; 14, 21)	(7, 3; 18, 8)	(2, 11; 13, 2)	(0, 10; 6, 10)	(4, 8; 21, 13)	(10, 12; 15, 11)
(13, 11; 17, 15)	(12, 7; 9, 1)	(7, 4; 23, 25)	(12, 9; 12, 5)	(4, 12; 18, 22)	(9, 5; 22, 25)	(1, 6; 25, 17)	(2, 0; 22, 15)	(7, 13; 21, 2)
(13, 10; 5, 18)	(6, 10; 4, 9)	(6, 3; 24, 10)	(1, 11; 18, 24)	(3, 0; 25, 9)	(4, 1; 15, 16)	(13, 0; 16, 12)	(8, 0; 14, 4)	(1, 13; 6, 23)
(5, 3; 20, 0)	(12, 1; 8, 7)	(4, 9; 0, 6)	(9, 3; 23, 15)	(8, 1; 9, 5)	(13, 6; 11, 22)	(6, 0; 21, 7)	(0, 9; 11, 24)	(5, 2; 12, 17)
(8, 11; 25, 11)	(2, 13; 3, 9)	(12, 8; 20, 3)	(6, 11; 16, 0)	(5, 12; 2, 4)	(12, 3; 21, 16)	(9, 2; 21, 1)	(10, 3; 1, 13)	(7, 8; 10, 7)
(7, 11; 3, 6)	(7, 10; 17, 0)	(0, 7; 20, 19)	(11, 4; 12, 4)	(9, 11; 7, 9)	(1, 2; 19, 10)	(3, 8; 22, 2)	(10, 5; 3, 16)	(13, 4; 20, 1)
(4, 2; 24, 14)	(5, 13; 13, 7)	(8, 5; 6, 15)	(9, 13; 10, 4)	(6, 4; 5, 2)	(6, 12; 19, 6)	(8, 6; 1, 23)	(9, 1; 13, 20)	(0, 5; 23, 18)
(8, 10; 24, 12)	(11, 0; 5, 8)	(13, 3; 14, 19)	(7, 5; 24, 5)	(11, 12; 10, 14)	(7, 1; 22, 4)	(11, 5; 1, 19)	(12, 0; 13, 17)	(3, 4; 17, 3)
(13, 8; 8, 0)	(10, 4; 7, 19)	(2, 7; 11, 16)	(2, 6; 20, 18)	(2, 10; 8, 25)	(6, 9; 3, 8)			

$a = 7$ :

(9, 6; 0, 9)	(5, 12; 4, 2)	(3, 12; 8, 14)	(6, 1; 20, 7)	(0, 11; 26, 18)	(2, 6; 3, 11)	(8, 13; 1, 12)	(4, 11; 13, 17)	(3, 8; 22, 24)
(5, 1; 26, 12)	(7, 4; 27, 18)	(4, 1; 19, 23)	(7, 12; 23, 1)	(4, 8; 15, 26)	(10, 0; 9, 6)	(9, 11; 12, 2)	(1, 12; 17, 21)	(12, 2; 22, 19)
(6, 0; 17, 25)	(13, 6; 22, 2)	(3, 13; 0, 18)	(0, 13; 8, 20)	(7, 3; 16, 2)	(5, 6; 19, 1)	(3, 0; 10, 13)	(13, 10; 19, 4)	(12, 9; 7, 10)
(6, 4; 4, 24)	(8, 7; 8, 6)	(9, 4; 6, 25)	(6, 11; 27, 8)	(2, 1; 18, 15)	(1, 11; 16, 4)	(4, 3; 20, 1)	(5, 0; 27, 15)	(7, 9; 26, 4)
(3, 5; 21, 25)	(1, 7; 11, 22)	(8, 6; 5, 21)	(7, 5; 20, 17)	(5, 8; 3, 23)	(13, 2; 17, 14)	(0, 7; 19, 24)	(6, 10; 16, 10)	(13, 7; 21, 9)
(5, 13; 7, 16)	(11, 3; 19, 11)	(5, 9; 22, 5)	(9, 2; 27, 20)	(0, 2; 16, 23)	(6, 3; 23, 26)	(6, 12; 6, 18)	(8, 12; 13, 20)	(2, 5; 13, 0)
(1, 13; 6, 13)	(12, 10; 11, 15)	(10, 3; 17, 12)	(2, 7; 10, 25)	(11, 5; 14, 6)	(12, 11; 3, 9)	(11, 8; 25, 7)	(2, 10; 26, 8)	(1, 10; 5, 25)
(0, 8; 11, 4)	(8, 2; 9, 2)	(2, 11; 24, 1)	(3, 9; 15, 3)	(8, 10; 0, 27)	(9, 0; 14, 21)	(0, 4; 7, 22)	(8, 1; 14, 10)	(10, 7; 3, 7)
(12, 0; 12, 5)	(9, 13; 11, 23)	(13, 11; 15, 10)	(2, 4; 21, 12)	(4, 13; 5, 3)	(1, 3; 27, 9)	(12, 4; 16, 0)	(4, 10; 2, 14)	(7, 11; 0, 5)
(10, 5; 18, 24)	(1, 9; 8, 24)	(9, 10; 1, 13)						

■

**Lemma 8.11:** There exists an SFS of type  $(4, 2)^a(2, 2)^{8-a}$  for each  $a \in \{0, 1, \dots, 8\}$ .

*Proof:* Let  $V = I_{16}$  and  $S = I_{16+2a}$ .  $V$  can be partitioned as  $V = \cup_{i=0}^7 \{2i, 2i+1\}$  and  $S$  can be partitioned as  $S = (\cup_{i=0}^{a-1} \{4i, 4i+1, 4i+2, 4i+3\}) \cup (\cup_{i=a}^7 \{2i, 2i+1\})$ . The required SFSs are presented as follows.

$a = 0$ :

(1, 9; 3, 14)	(14, 7; 9, 1)	(13, 11; 5, 14)	(8, 7; 13, 14)	(13, 5; 8, 7)	(1, 3; 5, 15)	(1, 12; 4, 2)	(7, 11; 15, 12)	(7, 15; 10, 8)
(6, 8; 12, 4)	(0, 5; 13, 3)	(14, 6; 5, 8)	(8, 2; 5, 7)	(10, 15; 0, 7)	(6, 12; 10, 0)	(11, 3; 13, 1)	(5, 8; 11, 15)	(6, 11; 2, 9)
(3, 15; 4, 6)	(8, 13; 1, 6)	(9, 10; 5, 2)	(8, 0; 10, 2)	(15, 12; 5, 1)	(14, 9; 13, 0)	(9, 11; 4, 7)	(3, 13; 0, 9)	(13, 9; 15, 10)
(1, 14; 12, 7)	(3, 4; 10, 12)	(13, 7; 4, 3)	(4, 8; 3, 0)	(12, 5; 14, 6)	(0, 4; 7, 14)	(9, 0; 6, 12)	(5, 10; 1, 12)	(12, 3; 7, 11)
(4, 1; 6, 8)	(15, 4; 13, 2)	(12, 11; 8, 3)	(15, 2; 12, 9)	(7, 5; 0, 2)	(1, 5; 9, 10)	(13, 14; 2, 11)	(9, 4; 11, 1)	(4, 12; 15, 9)
(10, 14; 3, 6)	(10, 6; 15, 13)	(14, 2; 10, 4)	(2, 6; 1, 14)	(0, 10; 4, 9)	(10, 3; 8, 14)	(6, 15; 11, 3)	(2, 11; 0, 6)	(0, 2; 8, 15)
(0, 7; 5, 11)	(1, 2; 11, 13)							

$a = 1$ :

(4, 14; 1, 5)	(14, 0; 6, 11)	(13, 2; 2, 6)	(6, 9; 0, 15)	(1, 13; 5, 13)	(7, 5; 17, 0)	(12, 4; 4, 2)	(5, 1; 14, 10)	(14, 7; 13, 10)
(8, 6; 2, 13)	(4, 3; 12, 14)	(8, 3; 16, 6)	(7, 13; 3, 11)	(6, 4; 3, 10)	(6, 11; 5, 11)	(2, 8; 7, 0)	(8, 1; 8, 4)	(1, 3; 9, 7)
(12, 15; 13, 0)	(6, 12; 12, 7)	(12, 11; 6, 3)	(10, 14; 15, 4)	(9, 4; 13, 8)	(3, 5; 11, 1)	(0, 2; 10, 9)	(1, 4; 17, 11)	(13, 6; 1, 16)
(13, 10; 8, 7)	(5, 9; 5, 2)	(2, 12; 11, 8)	(15, 6; 4, 6)	(3, 13; 10, 0)	(12, 10; 12, 5)	(6, 10; 17, 14)	(11, 14; 9, 0)	(1, 9; 6, 12)
(15, 5; 8, 3)	(0, 8; 5, 14)	(8, 12; 17, 1)	(10, 8; 3, 9)	(2, 11; 15, 17)	(11, 0; 16, 8)	(13, 0; 17, 12)	(5, 13; 4, 9)	(3, 14; 8, 2)
(11, 7; 14, 2)	(9, 14; 7, 14)	(9, 11; 1, 4)	(4, 15; 9, 15)	(10, 7; 6, 1)	(10, 15; 2, 11)	(9, 3; 17, 3)	(12, 9; 9, 16)	
(0, 7; 7, 4)	(15, 11; 10, 7)	(2, 15; 1, 14)	(3, 0; 13, 15)	(14, 2; 12, 3)	(5, 2; 13, 16)	(7, 15; 5, 12)	(1, 7; 15, 16)	(5, 8; 12, 15)

$a = 2$ :

(14, 1; 8, 7)	(1, 8; 11, 14)	(12, 9; 8, 14)	(3, 15; 3, 12)	(0, 10; 10, 5)	(4, 13; 10, 19)	(3, 9; 19, 11)	(14, 13; 3, 6)	(6, 13; 7, 2)
(4, 10; 16, 1)	(11, 1; 9, 16)	(4, 8; 4, 3)	(13, 11; 1, 5)	(0, 4; 13, 6)	(0, 8; 19, 15)	(1, 4; 15, 17)	(0, 5; 7, 11)	(11, 6; 3, 13)
(13, 9; 9, 15)	(7, 10; 3, 9)	(7, 4; 14, 0)	(13, 3; 8, 0)	(2, 12; 18, 3)	(15, 10; 4, 2)	(7, 11; 19, 8)	(14, 3; 17, 13)	(9, 6; 1, 4)
(11, 14; 4, 10)	(15, 12; 15, 10)	(11, 15; 6, 17)	(11, 2; 0, 11)	(9, 15; 0, 5)	(13, 5; 14, 12)	(14, 10; 11, 12)	(2, 15; 14, 9)	(12, 3; 9, 1)
(6, 10; 17, 19)	(14, 5; 15, 1)	(4, 11; 7, 12)	(7, 2; 1, 13)	(8, 14; 0, 9)	(14, 6; 16, 14)	(4, 6; 5, 18)	(12, 1; 5, 13)	(7, 3; 15, 16)
(6, 0; 9, 12)	(9, 2; 2, 17)	(9, 7; 18, 7)	(2, 8; 16, 10)	(15, 13; 11, 13)	(0, 15; 8, 16)	(10, 12; 0, 7)	(10, 5; 13, 18)	(14, 7; 5, 2)
(11, 8; 2, 18)	(5, 6; 6, 0)	(7, 0; 4, 17)	(5, 9; 16, 3)	(1, 9; 10, 6)	(8, 10; 6, 8)	(6, 2; 8, 15)	(7, 12; 6, 12)	(1, 13; 4, 18)
(15, 8; 1, 7)	(1, 2; 19, 12)	(0, 3; 18, 14)	(8, 5; 17, 5)	(12, 5; 4, 19)	(3, 5; 2, 10)	(4, 12; 2, 11)		

$a = 3$ :

(15, 13; 12, 9)	(11, 14; 7, 11)	(5, 10; 13, 6)	(7, 14; 0, 4)	(9, 0; 9, 19)	(10, 1; 8, 21)	(7, 11; 18, 9)	(12, 3; 1, 9)	(12, 9; 20, 0)
(9, 10; 1, 12)	(15, 0; 15, 17)	(2, 8; 3, 9)	(4, 6; 14, 16)	(5, 8; 12, 2)	(3, 7; 2, 10)	(1, 9; 4, 13)	(6, 15; 19, 11)	(15, 12; 13, 16)
(7, 5; 1, 21)	(8, 12; 5, 21)	(8, 13; 13, 17)	(13, 11; 1, 5)	(3, 10; 11, 20)	(5, 13; 20, 4)	(12, 2; 11, 2)	(0, 12; 10, 4)	(10, 0; 5, 18)
(6, 0; 7, 21)	(6, 13; 6, 0)	(6, 1; 9, 5)	(13, 2; 21, 16)	(15, 11; 2, 4)	(2, 4; 13, 18)	(9, 13; 3, 11)	(6, 9; 8, 17)	(8, 10; 7, 19)
(1, 7; 15, 11)	(15, 8; 8, 0)	(5, 14; 5, 17)	(2, 11; 12, 14)	(0, 3; 16, 12)	(14, 6; 18, 2)	(12, 7; 3, 8)	(1, 14; 6, 12)	(10, 13; 15, 2)
(8, 0; 11, 6)	(3, 5; 15, 0)	(5, 1; 19, 16)	(1, 13; 7, 10)	(3, 14; 3, 13)	(15, 5; 18, 3)	(4, 10; 4, 3)	(3, 1; 14, 17)	(9, 7; 7, 16)
(4, 14; 19, 15)	(8, 14; 16, 10)	(0, 13; 8, 14)	(7, 15; 5, 14)	(11, 3; 8, 19)	(11, 4; 21, 0)	(4, 15; 1, 7)	(7, 4; 6, 20)	(4, 9; 2, 5)
(4, 12; 17, 12)	(9, 3; 18, 21)	(8, 6; 1, 4)	(12, 5; 14, 7)	(10, 2; 10, 0)	(14, 2; 8, 1)	(15, 9; 10, 6)	(7, 2; 19, 17)	(8, 1; 20, 18)
(11, 12; 6, 15)	(14, 10; 14, 9)	(11, 0; 13, 20)	(2, 6; 15, 20)	(6, 11; 3, 10)				

$a = 4$ :

(0, 8; 14, 7)	(7, 1; 6, 10)	(2, 14; 17, 0)	(12, 11; 14, 22)	(2, 4; 12, 21)	(6, 9; 3, 23)	(13, 9; 8, 1)	(5, 3; 20, 3)	(15, 6; 20, 6)
(8, 2; 2, 15)	(11, 5; 2, 23)	(3, 7; 8, 19)	(2, 9; 9, 18)	(7, 2; 3, 22)	(9, 4; 5, 0)	(8, 1; 22, 9)	(6, 1; 4, 19)	(13, 6; 7, 18)
(15, 8; 3, 10)	(10, 9; 11, 15)	(8, 13; 23, 11)	(12, 3; 15, 1)	(15, 12; 2, 9)	(11, 4; 20, 15)	(5, 10; 1, 16)	(1, 12; 18, 23)	(8, 11; 0, 8)
(4, 14; 16, 2)	(12, 5; 12, 6)	(4, 7; 18, 1)	(5, 9; 19, 22)	(2, 15; 1, 19)	(1, 15; 15, 16)	(10, 3; 0, 22)	(15, 13; 14, 0)	(12, 6; 10, 0)
(3, 14; 18, 11)	(11, 6; 21, 9)	(13, 10; 2, 13)	(4, 8; 6, 19)	(4, 13; 4, 22)	(13, 1; 12, 5)	(0, 12; 8, 4)	(9, 11; 7, 13)	(12, 10; 5, 17)
(0, 11; 10, 16)	(8, 3; 13, 21)	(5, 15; 17, 13)	(0, 5; 5, 15)	(15, 10; 7, 8)	(4, 3; 23, 14)	(4, 1; 17, 7)	(7, 5; 0, 7)	(10, 2; 10, 14)
(0, 10; 21, 6)	(6, 0; 11, 22)	(11, 14; 1, 12)	(8, 10; 20, 12)	(8, 6; 5, 1)	(14, 1; 13, 8)	(0, 13; 19, 17)	(0, 14; 9, 20)	(13, 11; 6, 3)
(9, 7; 20, 2)	(15, 0; 12, 18)	(4, 12; 3, 13)	(2, 1; 11, 20)	(1, 5; 21, 14)	(13, 14; 10, 15)	(11, 7; 4, 17)	(14, 12; 7, 19)	(6, 2; 8, 16)
(3, 9; 10, 12)	(10, 14; 4, 3)	(6, 3; 17, 2)	(15, 11; 11, 5)	(0, 2; 23, 13)	(14, 7; 21, 5)	(5, 8; 4, 18)	(9, 15; 21, 4)	(9, 14; 6, 14)
(3, 13; 9, 16)	(7, 10; 9, 23)	(7, 12; 11, 16)						

$a = 5$ :

(14, 9; 22, 15)	(1, 3; 10, 19)	(4, 2; 16, 1)	(7, 13; 5, 9)	(7, 0; 16, 7)	(14, 10; 12, 2)	(1, 11; 5, 15)	(12, 10; 1, 5)	(5, 1; 7, 23)
(10, 8; 7, 14)	(8, 13; 11, 20)	(5, 8; 3, 6)	(14, 8; 9, 0)	(3, 11; 25, 3)	(10, 9; 8, 3)	(15, 12; 0, 10)	(4, 0; 23, 19)	(13, 10; 13, 25)
(5, 3; 22, 16)	(4, 12; 7, 25)	(2, 7; 8, 19)	(10, 7; 6, 23)	(1, 10; 9, 16)	(3, 6; 18, 11)	(14, 2; 11, 14)	(12, 9; 12, 11)	(3, 9; 13, 23)
(8, 11; 22, 2)	(0, 3; 9, 14)	(12, 5; 15, 2)	(2, 10; 10, 18)	(6, 2; 20, 9)	(3, 13; 0, 17)	(4, 1; 12, 6)	(10, 3; 24, 15)	(9, 7; 4, 25)
(5, 15; 4, 19)	(14, 11; 23, 18)	(14, 4; 13, 5)	(1, 13; 8, 18)	(5, 6; 21, 25)	(9, 2; 2, 21)	(1, 15; 11, 13)	(1, 14; 17, 21)	(6, 15; 22, 5)
(4, 13; 3, 15)	(15, 13; 21, 7)	(8, 1; 4, 24)	(13, 0; 10, 24)	(8, 0; 25, 5)	(7, 14; 10, 20)	(0, 6; 8, 4)	(9, 15; 9, 6)	(8, 15; 1, 15)
(4, 3; 20, 2)	(6, 10; 0, 19)	(11, 6; 1, 6)	(10, 4; 4, 17)	(4, 15; 18, 14)	(12, 14; 16, 4)	(5, 7; 1, 24)	(5, 2; 0, 12)	(12, 6; 17, 24)
(1, 2; 25, 22)	(0, 15; 20, 12)	(15, 7; 2, 17)	(11, 13; 4, 12)	(12, 1; 14, 20)	(3, 14; 1, 8)	(15, 2; 23, 3)	(11, 15; 8, 16)	(4, 9; 0, 24)
(14, 6; 3, 7)	(7, 11; 11, 0)	(2, 0; 17, 15)	(6, 13; 2, 16)	(11, 2; 13, 24)	(11, 12; 19, 9)	(9, 11; 10, 7)	(12, 7; 3, 18)	(5, 0; 13, 18)
(5, 9; 5, 20)	(5, 11; 17, 14)	(7, 4; 22, 21)	(13, 9; 14, 1)	(6, 8; 23, 10)	(14, 13; 19, 6)	(12, 0; 6, 21)	(0, 10; 11, 22)	(3, 8; 12, 21)
(8, 12; 8, 13)								

$a = 6$ :

(10, 2; 10, 13)	(14, 12; 18, 9)	(9, 14; 10, 6)	(3, 11; 10, 1)	(14, 5; 16, 21)	(12, 8; 8, 13)	(4, 2; 15, 0)	(10, 7; 3, 16)	(3, 1; 17, 21)
(9, 6; 11, 21)	(14, 7; 0, 20)	(7, 9; 7, 8)	(2, 14; 14, 23)	(8, 13; 11, 5)	(15, 0; 8, 22)	(5, 2; 24, 3)	(2, 9; 22, 25)	(13, 14; 15, 2)
(12, 15; 14, 21)	(3, 14; 8, 19)	(13, 4; 19, 21)	(11, 15; 9, 2)	(5, 7; 25, 27)	(13, 10; 4, 8)	(1, 5; 20, 13)	(4, 14; 25, 4)	(4, 12; 7, 20)
(5, 0; 6, 15)	(12, 3; 16, 0)	(2, 15; 18, 11)	(15, 8; 3, 6)	(13, 11; 6, 13)	(3, 13; 27, 12)	(7, 2; 9, 21)	(1, 11; 26, 25)	(5, 10; 5, 0)
(6, 0; 25, 18)	(7, 1; 11, 23)	(1, 14; 12, 24)	(6, 10; 6, 24)	(13, 1; 16, 14)	(11, 9; 12, 0)	(9, 4; 5, 26)	(15, 9; 24, 4)	(9, 10; 27, 2)
(2, 8; 20, 2)	(6, 11; 16, 8)	(8, 7; 10, 26)	(0, 8; 21, 27)	(12, 6; 27, 3)	(0, 7; 4, 19)	(0, 11; 7, 14)	(1, 15; 10, 15)	(14, 8; 22, 7)
(13, 15; 0, 7)	(6, 1; 9, 5)	(5, 11; 18, 4)	(4, 0; 23, 24)	(14, 11; 5, 3)	(3, 7; 2, 24)	(12, 1; 22, 4)	(12, 7; 6, 1)	(9, 3; 13, 9)
(2, 13; 1, 17)	(3, 4; 3, 14)	(12, 0; 5, 10)	(8, 11; 15, 24)	(6, 13; 23, 10)	(8, 4; 12, 1)	(4, 6; 22, 2)	(0, 2; 12, 16)	(13, 9; 3, 20)
(12, 10; 26, 17)	(10, 14; 11, 1)	(14, 0; 13, 17)	(6, 5; 7, 17)	(15, 7; 5, 17)	(3, 0; 11, 20)	(11, 4; 27, 17)	(0, 13; 9, 26)	(12, 11; 19, 11)
(9, 5; 14, 1)	(5, 12; 2, 12)	(6, 15; 20, 1)	(13, 7; 22, 18)	(10, 3; 15, 18)	(8, 10; 14, 9)	(4, 1; 6, 18)	(3, 8; 25, 23)	(6, 2; 19, 26)
(8, 6; 0, 4)	(15, 4; 13, 16)	(9, 12; 15, 23)	(2, 1; 27, 8)	(3, 5; 26, 22)	(10, 15; 25, 12)	(1, 10; 7, 19)	(5, 15; 19, 23)	

$a = 7$ :

(3, 12; 16, 8)	(15, 8; 15, 3)	(11, 5; 13, 2)	(7, 13; 6, 9)	(5, 8; 0, 6)	(9, 7; 29, 2)	(4, 1; 7, 28)	(10, 9; 11, 4)	(4, 14; 27, 3)
(1, 15; 18, 6)	(8, 13; 11, 23)	(4, 12; 6, 15)	(2, 14; 2, 18)	(15, 7; 23, 8)	(3, 11; 15, 28)	(2, 5; 14, 29)	(4, 6; 19, 24)	(10, 14; 24, 14)
(1, 10; 17, 26)	(11, 8; 12, 9)	(0, 8; 14, 26)	(0, 9; 8, 21)	(4, 10; 5, 13)	(10, 6; 8, 0)	(0, 6; 11, 22)	(9, 2; 26, 28)	(12, 8; 2, 4)
(0, 15; 16, 10)	(3, 7; 17, 25)	(10, 8; 7, 1)	(5, 0; 28, 24)	(3, 13; 29, 1)	(15, 4; 26, 4)	(14, 1; 21, 11)	(9, 15; 7, 24)	(12, 10; 28, 18)
(2, 7; 24, 11)	(4, 3; 0, 21)	(4, 7; 16, 1)	(14, 8; 25, 8)	(0, 7; 27, 7)	(9, 3; 10, 13)	(2, 0; 15, 20)	(9, 1; 9, 15)	(7, 12; 3, 21)
(1, 8; 29, 13)	(13, 1; 12, 8)	(13, 0; 13, 19)	(11, 7; 26, 0)	(15, 13; 22, 14)	(0, 11; 5, 25)	(9, 6; 6, 23)	(3, 10; 9, 2)	(1, 7; 4, 19)
(4, 2; 12, 22)	(15, 2; 0, 13)	(11, 14; 6, 19)	(10, 13; 15, 10)	(8, 3; 24, 22)	(4, 11; 17, 29)	(5, 1; 20, 5)	(5, 10; 25, 3)	(7, 14; 10, 5)
(14, 0; 4, 9)	(2, 6; 17, 9)	(2, 11; 8, 3)	(15, 10; 19, 12)	(5, 7; 22, 18)	(12, 14; 23, 13)	(12, 9; 20, 0)	(11, 1; 16, 24)	(10, 0; 6, 29)
(12, 11; 14, 11)	(6, 8; 27, 5)	(9, 13; 5, 3)	(14, 6; 20, 16)	(1, 3; 27, 14)	(9, 4; 25, 14)	(5, 6; 1, 26)	(3, 6; 18, 3)	(3, 15; 11, 20)
(1, 12; 22, 10)	(15, 6; 25, 2)	(15, 5; 21, 17)	(13, 4; 2, 20)	(13, 11; 7, 18)	(12, 2; 19, 1)	(7, 8; 20, 28)	(6, 12; 7, 29)	(3, 5; 23, 19)
(2, 8; 10, 21)	(12, 0; 12, 17)	(13, 14; 0, 17)	(6, 13; 21, 28)	(4, 0; 18, 23)	(2, 1; 23, 25)	(14, 9; 22, 1)	(13, 5; 4, 16)	(12, 15; 9, 5)
(10, 2; 27, 16)	(11, 15; 27, 1)	(11, 6; 10, 4)	(5, 14; 15, 7)	(3, 14; 12, 26)	(5, 9; 12, 27)			

$a = 8$ :

(15, 7; 25, 22)	(2, 7; 26, 30)	(2, 0; 21, 25)	(7, 10; 31, 9)	(3, 13; 8, 19)	(5, 7; 20, 3)	(6, 5; 2, 7)	(12, 5; 21, 16)	(6, 10; 25, 19)
(1, 4; 4, 25)	(13, 7; 17, 0)	(5, 2; 1, 14)	(7, 12; 4, 29)	(4, 11; 19, 28)	(14, 4; 6, 22)	(5, 10; 30, 17)	(10, 13; 18, 1)	(1, 8; 9, 6)
(5, 14; 4, 26)	(4, 6; 31, 18)	(7, 9; 6, 1)	(14, 0; 9, 12)	(6, 3; 1, 9)	(0, 8; 13, 22)	(6, 2; 27, 10)	(11, 13; 9, 3)	(12, 3; 13, 23)
(15, 3; 12, 18)	(8, 12; 30, 0)	(15, 1; 21, 27)	(14, 10; 8, 5)	(4, 2; 15, 29)	(12, 11; 2, 18)	(9, 12; 9, 5)	(11, 15; 24, 5)	(15, 4; 0, 26)
(9, 3; 22, 24)	(5, 8; 25, 28)	(8, 2; 2, 11)	(14, 12; 7, 17)	(3, 11; 31, 25)	(2, 11; 13, 17)	(0, 6; 29, 24)	(3, 0; 30, 10)	(8, 15; 1, 7)
(15, 0; 17, 4)	(9, 11; 0, 11)	(10, 8; 10, 29)	(10, 2; 24, 0)	(9, 6; 23, 26)	(11, 7; 7, 10)	(12, 10; 6, 14)	(15, 13; 23, 2)	(0, 7; 18, 28)
(15, 9; 14, 10)	(12, 1; 10, 12)	(8, 3; 26, 20)	(13, 9; 21, 29)	(2, 12; 22, 3)	(3, 10; 3, 28)	(6, 12; 11, 28)	(8, 4; 14, 23)	(10, 1; 15, 26)
(13, 14; 10, 15)	(13, 8; 31, 12)	(13, 4; 13, 7)	(9, 4; 30, 3)	(1, 11; 8, 30)	(9, 5; 27, 31)	(0, 12; 31, 8)	(6, 14; 0, 20)	(0, 9; 20, 15)
(4, 3; 17, 27)	(0, 5; 19, 23)	(6, 15; 3, 8)	(6, 13; 30, 4)	(11, 8; 4, 15)	(14, 8; 24, 3)	(9, 10; 13, 4)	(10, 15; 16, 11)	(2, 13; 28, 16)
(4, 0; 5, 16)	(2, 15; 20, 9)	(1, 2; 31, 19)	(7, 8; 8, 27)	(5, 13; 22, 5)	(7, 1; 5, 23)	(6, 11; 16, 6)	(7, 14; 19, 11)	(15, 12; 19, 15)
(11, 14; 27, 1)	(10, 0; 7, 27)	(14, 2; 23, 18)	(11, 5; 29, 12)	(7, 3; 16, 2)	(1, 6; 22, 17)	(5, 3; 0, 15)	(3, 1; 29, 11)	(3, 14; 21, 14)
(4, 10; 12, 2)	(4, 12; 20, 1)	(4, 7; 21, 24)	(0, 13; 11, 6)	(9, 14; 25, 2)	(9, 2; 12, 8)	(11, 0; 14, 26)	(8, 6; 21, 5)	(13, 1; 20, 14)
(1, 5; 18, 24)	(1, 9; 7, 28)	(1, 14; 13, 16)	(5, 15; 6, 13)					

■

**Lemma 8.12:** There exists an SFS of type  $(4, 2)^a(2, 2)^{9-a}$  for each  $a \in \{0, 1, \dots, 9\}$ .

*Proof:* Let  $V = I_{18}$  and  $S = I_{18+2a}$ .  $V$  can be partitioned as  $V = \cup_{i=0}^8 \{2i, 2i+1\}$  and  $S$  can be partitioned as  $S = (\cup_{i=0}^{a-1} \{4i, 4i+1, 4i+2, 4i+3\}) \cup (\cup_{i=a}^8 \{2i, 2i+1\})$ . The required SFSs are presented as follows.

$a = 0$ :

(7, 4; 13, 1)	(9, 6; 1, 14)	(10, 12; 3, 9)	(9, 17; 0, 3)	(9, 0; 4, 15)	(10, 17; 5, 7)	(0, 3; 5, 10)	(0, 17; 11, 13)	(13, 5; 11, 9)
(12, 15; 1, 7)	(15, 5; 3, 12)	(14, 16; 3, 4)	(17, 15; 9, 2)	(5, 0; 17, 14)	(12, 3; 16, 14)	(11, 4; 3, 14)	(16, 1; 12, 14)	(3, 1; 7, 17)
(10, 7; 2, 14)	(9, 14; 12, 5)	(5, 10; 0, 15)	(1, 12; 8, 4)	(14, 2; 6, 1)	(15, 4; 16, 8)	(8, 6; 16, 12)	(2, 12; 15, 5)	(8, 13; 4, 14)
(7, 13; 8, 10)	(16, 15; 0, 5)	(14, 10; 17, 8)	(15, 2; 4, 11)	(13, 2; 0, 7)	(8, 0; 3, 6)	(1, 4; 11, 15)	(13, 6; 3, 17)	(4, 3; 0, 6)
(9, 12; 11, 17)	(11, 8; 0, 2)	(13, 10; 16, 6)	(0, 4; 9, 7)	(10, 6; 4, 13)	(14, 1; 9, 13)	(14, 0; 16, 2)	(2, 5; 16, 12)	(10, 3; 1, 12)
(16, 8; 11, 1)	(2, 4; 17, 12)	(7, 1; 16, 3)	(5, 12; 6, 2)	(8, 7; 17, 5)	(7, 14; 11, 0)	(13, 1; 2, 5)	(11, 17; 6, 4)	(17, 5; 1, 10)
(16, 6; 15, 2)	(8, 14; 7, 10)	(6, 12; 0, 10)	(17, 2; 8, 14)	(7, 17; 12, 15)	(4, 9; 2, 10)	(15, 11; 13, 17)	(8, 3; 13, 15)	(11, 13; 1, 15)
(6, 11; 5, 9)	(6, 3; 8, 11)	(16, 2; 9, 10)	(3, 7; 4, 9)	(9, 16; 13, 6)	(15, 1; 10, 6)	(5, 16; 7, 8)	(0, 11; 8, 12)	(9, 11; 7, 16)

$a = 1$ :

(4, 11; 5, 1)	(7, 4; 10, 15)	(14, 4; 9, 18)	(9, 3; 13, 0)	(2, 1; 17, 14)	(11, 8; 0, 7)	(4, 0; 19, 14)	(6, 12; 16, 5)	(8, 10; 14, 5)
(11, 16; 6, 15)	(15, 17; 4, 9)	(6, 13; 17, 7)	(7, 17; 16, 7)	(11, 1; 10, 4)	(12, 11; 17, 3)	(4, 2; 16, 0)	(16, 3; 17, 2)	(5, 15; 8, 0)
(2, 14; 3, 6)	(14, 11; 11, 2)	(15, 7; 1, 12)	(0, 8; 6, 16)	(1, 7; 18, 13)	(1, 5; 9, 5)	(14, 3; 19, 10)	(0, 14; 12, 4)	(13, 11; 9, 19)
(17, 3; 1, 15)	(12, 8; 9, 2)	(14, 5; 1, 14)	(11, 3; 18, 8)	(1, 15; 19, 6)	(16, 10; 1, 9)	(13, 8; 1, 8)	(9, 12; 1, 19)	(5, 0; 15, 17)
(14, 16; 8, 13)	(6, 16; 10, 0)	(4, 17; 17, 11)	(3, 6; 12, 6)	(0, 15; 5, 13)	(10, 9; 6, 17)	(9, 11; 16, 14)	(16, 13; 5, 11)	(16, 7; 14, 3)
(6, 4; 4, 2)	(3, 10; 3, 16)	(6, 10; 15, 19)	(5, 8; 12, 19)	(0, 13; 10, 18)	(7, 13; 6, 2)	(6, 15; 3, 11)	(13, 9; 3, 12)	(2, 12; 13, 10)
(12, 17; 0, 6)	(8, 15; 18, 15)	(0, 2; 7, 8)	(10, 15; 10, 2)	(15, 3; 14, 7)	(17, 9; 5, 8)	(14, 1; 7, 15)	(4, 8; 13, 3)	(14, 7; 5, 0)
(7, 2; 19, 11)	(9, 16; 4, 7)	(9, 2; 15, 9)	(2, 6; 18, 1)	(8, 7; 4, 17)	(12, 10; 18, 7)	(3, 0; 9, 11)	(9, 5; 2, 18)	(6, 17; 13, 14)
(17, 5; 3, 10)	(2, 17; 12, 2)	(5, 13; 13, 16)	(10, 13; 0, 4)	(1, 16; 16, 12)	(12, 4; 8, 12)	(1, 10; 8, 11)	(5, 12; 4, 11)	

$a = 2$ :

(12, 4; 21, 15)	(8, 1; 8, 17)	(10, 3; 1, 13)	(1, 7; 7, 21)	(4, 2; 16, 18)	(12, 16; 7, 2)	(9, 15; 8, 4)	(12, 7; 14, 1)	(12, 10; 6, 3)
(12, 1; 10, 4)	(14, 8; 11, 16)	(2, 7; 3, 19)	(17, 0; 9, 7)	(0, 7; 5, 8)	(16, 13; 11, 3)	(15, 17; 3, 14)	(14, 12; 5, 9)	(1, 14; 6, 20)
(1, 13; 5, 14)	(6, 15; 15, 12)	(10, 4; 19, 7)	(0, 13; 18, 12)	(5, 6; 3, 18)	(4, 8; 20, 3)	(11, 6; 20, 5)	(11, 16; 4, 19)	(5, 12; 19, 13)
(4, 3; 14, 12)	(16, 9; 17, 14)	(14, 3; 21, 3)	(6, 3; 17, 0)	(8, 0; 4, 14)	(11, 9; 7, 3)	(2, 8; 9, 2)	(17, 9; 5, 2)	(0, 10; 17, 11)
(17, 6; 16, 4)	(3, 12; 8, 18)	(8, 5; 5, 21)	(8, 15; 7, 10)	(16, 1; 16, 13)	(17, 14; 15, 17)	(10, 5; 20, 4)	(15, 4; 1, 5)	(9, 7; 9, 18)
(11, 15; 16, 9)	(2, 13; 20, 0)	(5, 2; 12, 11)	(17, 12; 12, 0)	(16, 7; 6, 12)	(16, 14; 8, 0)	(0, 2; 13, 15)	(2, 15; 17, 21)	(6, 4; 2, 6)
(17, 2; 10, 1)	(12, 9; 20, 11)	(17, 8; 6, 19)	(7, 15; 13, 20)	(11, 14; 2, 12)	(0, 3; 16, 20)	(5, 11; 10, 17)	(13, 11; 6, 21)	(9, 10; 21, 0)
(4, 7; 4, 17)	(10, 1; 12, 9)	(0, 6; 19, 21)	(13, 6; 7, 13)	(9, 5; 1, 16)	(11, 17; 13, 8)	(0, 9; 10, 6)	(7, 10; 2, 16)	(9, 1; 19, 15)
(10, 13; 8, 10)	(16, 10; 5, 18)	(3, 16; 10, 15)	(5, 14; 14, 7)	(1, 17; 11, 18)	(2, 6; 14, 8)	(3, 13; 19, 9)	(15, 5; 6, 0)	(4, 14; 13, 10)
(15, 3; 11, 2)	(6, 16; 9, 1)	(5, 13; 2, 15)	(13, 14; 1, 4)	(8, 11; 1, 18)	(4, 11; 0, 11)	(7, 8; 0, 15)		

$a = 3$ :

(8, 12; 2, 9)	(0, 14; 15, 8)	(17, 10; 8, 2)	(12, 1; 16, 15)	(3, 16; 17, 9)	(11, 1; 10, 12)	(8, 1; 7, 19)	(7, 9; 1, 10)	(17, 11; 15, 11)
(3, 5; 22, 1)	(10, 13; 13, 11)	(11, 9; 4, 2)	(15, 3; 0, 15)	(14, 9; 22, 7)	(0, 8; 18, 11)	(17, 8; 17, 3)	(11, 6; 22, 14)	(13, 16; 7, 15)
(16, 15; 11, 19)	(13, 14; 2, 10)	(3, 10; 12, 20)	(4, 15; 2, 14)	(12, 2; 0, 22)	(2, 10; 9, 15)	(15, 0; 6, 13)	(12, 16; 1, 6)	(16, 7; 2, 21)
(13, 0; 17, 22)	(3, 1; 13, 18)	(3, 6; 2, 11)	(15, 8; 8, 22)	(5, 1; 20, 14)	(4, 6; 20, 15)	(0, 11; 9, 20)	(17, 13; 20, 1)	(5, 17; 19, 13)
(14, 6; 16, 9)	(5, 2; 2, 12)	(3, 8; 10, 16)	(8, 11; 1, 5)	(0, 17; 7, 12)	(6, 13; 5, 0)	(12, 3; 21, 8)	(14, 12; 11, 4)	(11, 7; 19, 8)
(0, 10; 5, 14)	(9, 12; 20, 13)	(2, 16; 20, 8)	(13, 1; 4, 8)	(14, 16; 14, 12)	(16, 4; 13, 3)	(6, 9; 6, 8)	(13, 3; 14, 23)	(7, 17; 0, 4)
(6, 15; 10, 17)	(7, 13; 9, 3)	(0, 7; 16, 23)	(4, 17; 16, 21)	(7, 5; 5, 15)	(17, 12; 10, 14)	(15, 10; 7, 23)	(7, 2; 18, 14)	(4, 8; 23, 4)
(14, 4; 18, 1)	(5, 15; 3, 4)	(1, 17; 9, 6)	(6, 0; 21, 4)	(6, 10; 19, 1)	(5, 12; 17, 23)	(15, 17; 18, 5)	(12, 11; 3, 7)	(5, 6; 7, 18)
(9, 16; 5, 16)	(14, 11; 6, 23)	(2, 11; 21, 13)	(2, 15; 16, 1)	(14, 8; 13, 0)	(10, 5; 21, 0)	(6, 2; 3, 23)	(12, 4; 12, 5)	(0, 2; 19, 10)
(9, 1; 21, 23)	(9, 10; 3, 18)	(7, 1; 11, 22)	(15, 9; 9, 12)	(11, 16; 0, 18)	(14, 3; 3, 19)	(4, 9; 19, 0)	(10, 16; 10, 4)	(1, 14; 5, 17)
(8, 13; 12, 21)	(5, 13; 16, 6)	(2, 9; 11, 17)	(4, 7; 17, 7)	(4, 10; 6, 22)	(7, 8; 6, 20)			

$a = 4$ :

(13, 6; 11, 23)	(5, 10; 4, 23)	(17, 13; 22, 17)	(15, 12; 9, 18)	(9, 14; 10, 25)	(17, 0; 21, 5)	(13, 1; 13, 24)	(8, 13; 9, 25)	(4, 13; 2, 4)
(11, 16; 6, 14)	(4, 15; 12, 6)	(5, 7; 5, 25)	(7, 4; 24, 0)	(6, 10; 16, 10)	(1, 15; 8, 4)	(16, 0; 22, 4)	(3, 16; 11, 17)	(8, 0; 8, 13)
(14, 3; 20, 1)	(9, 10; 11, 5)	(14, 1; 5, 19)	(7, 15; 7, 19)	(17, 2; 20, 8)	(15, 10; 1, 25)	(17, 8; 23, 7)	(10, 12; 6, 13)	(4, 17; 1, 15)
(9, 7; 6, 2)	(2, 11; 25, 3)	(6, 9; 9, 4)	(1, 5; 6, 18)	(17, 9; 12, 18)	(6, 16; 18, 7)	(15, 5; 16, 21)	(17, 3; 9, 2)	(8, 7; 20, 10)
(3, 10; 14, 22)	(14, 16; 2, 21)	(6, 4; 22, 5)	(14, 13; 8, 14)	(9, 3; 15, 21)	(2, 16; 23, 0)	(14, 7; 17, 18)	(9, 0; 24, 20)	(6, 1; 17, 25)
(0, 4; 16, 18)	(11, 12; 2, 11)	(11, 6; 1, 24)	(14, 11; 13, 4)	(13, 11; 5, 16)	(2, 14; 16, 11)	(11, 7; 22, 21)	(4, 9; 3, 14)	(0, 15; 11, 14)
(12, 4; 17, 19)	(16, 12; 8, 5)	(14, 8; 6, 15)	(15, 2; 24, 17)	(6, 8; 21, 3)	(9, 5; 22, 1)	(15, 17; 3, 13)	(11, 5; 17, 12)	(16, 13; 12, 1)
(7, 10; 3, 8)	(12, 3; 10, 24)	(12, 0; 23, 25)	(10, 8; 2, 24)	(5, 2; 2, 13)	(10, 0; 17, 7)	(1, 2; 14, 10)	(10, 1; 20, 15)	(6, 3; 8, 0)
(11, 17; 0, 10)	(8, 2; 18, 22)	(5, 14; 7, 24)	(12, 17; 14, 16)	(14, 10; 0, 9)	(5, 16; 3, 20)	(1, 16; 9, 16)	(7, 3; 23, 16)	(6, 15; 20, 2)
(17, 7; 11, 4)	(15, 16; 15, 10)	(12, 8; 1, 4)	(10, 2; 21, 12)	(2, 7; 1, 9)	(3, 13; 3, 18)	(8, 15; 5, 0)	(9, 13; 7, 0)	(5, 12; 0, 15)
(2, 13; 15, 19)	(8, 5; 19, 14)	(9, 16; 19, 13)	(4, 3; 25, 13)	(11, 0; 15, 9)	(0, 13; 6, 10)	(4, 1; 23, 21)	(9, 11; 8, 23)	(17, 6; 6, 19)
(0, 3; 19, 12)	(4, 11; 7, 20)	(12, 1; 7, 22)	(1, 8; 11, 12)	(12, 14; 3, 12)				

$a = 5$ :

(16, 11; 22, 14)	(0, 2; 15, 16)	(6, 1; 10, 27)	(17, 10; 2, 25)	(2, 4; 27, 24)	(16, 12; 19, 20)	(16, 10; 9, 15)	(5, 14; 0, 23)	(10, 8; 13, 10)
(12, 10; 0, 16)	(7, 11; 26, 11)	(14, 10; 1, 5)	(7, 2; 2, 17)	(13, 3; 21, 24)	(10, 2; 14, 18)	(6, 12; 9, 5)	(15, 1; 14, 9)	(15, 10; 12, 23)
(0, 10; 17, 26)	(12, 4; 21, 14)	(6, 3; 25, 3)	(8, 6; 1, 11)	(6, 9; 24, 26)	(9, 5; 20, 22)	(12, 5; 15, 3)	(9, 7; 10, 6)	(17, 6; 18, 7)
(9, 12; 25, 12)	(11, 14; 4, 13)	(1, 8; 6, 12)	(13, 2; 12, 3)	(15, 11; 27, 16)	(4, 7; 7, 0)	(9, 16; 21, 8)	(15, 3; 20, 8)	(3, 17; 14, 11)
(13, 9; 0, 27)	(8, 16; 24, 5)	(4, 0; 25, 5)	(14, 4; 18, 3)	(14, 9; 14, 7)	(12, 7; 18, 8)	(15, 9; 11, 2)	(3, 4; 19, 12)	(2, 1; 26, 20)
(11, 3; 17, 1)	(14, 3; 16, 9)	(13, 4; 26, 13)	(0, 14; 19, 6)	(0, 8; 7, 27)	(6, 5; 16, 21)	(4, 11; 6, 15)	(16, 0; 11, 23)	(1, 13; 5, 19)
(2, 17; 23, 9)	(0, 9; 13, 9)	(10, 6; 22, 6)	(15, 16; 3, 4)	(11, 1; 23, 7)	(11, 12; 2, 24)	(0, 3; 22, 18)	(2, 14; 22, 11)	(7, 17; 20, 24)
(15, 12; 7, 26)	(8, 11; 3, 9)	(14, 6; 2, 8)	(0, 6; 20, 4)	(12, 2; 1, 10)	(10, 7; 19, 3)	(2, 11; 8, 0)	(15, 5; 6, 18)	(0, 17; 12, 8)
(13, 10; 11, 7)	(17, 9; 5, 3)	(4, 9; 4, 1)	(15, 6; 19, 0)	(9, 3; 23, 15)	(16, 3; 0, 10)	(17, 11; 10, 19)	(2, 8; 25, 21)	(7, 14; 21, 27)
(1, 4; 16, 22)	(17, 13; 16, 4)	(12, 1; 11, 4)	(5, 1; 17, 25)	(15, 7; 22, 5)	(14, 16; 17, 12)	(13, 16; 2, 6)	(12, 3; 27, 13)	(11, 5; 12, 5)
(3, 5; 2, 26)	(5, 0; 24, 14)	(13, 14; 10, 20)	(8, 17; 0, 22)	(17, 1; 21, 15)	(8, 7; 23, 4)	(5, 2; 19, 13)	(16, 1; 13, 18)	(13, 15; 15, 17)
(17, 15; 13, 1)	(12, 17; 17, 6)	(14, 8; 26, 15)	(4, 8; 20, 2)	(15, 0; 10, 21)	(1, 10; 24, 8)	(5, 10; 4, 27)	(13, 8; 14, 8)	(6, 4; 17, 23)
(11, 13; 18, 25)	(7, 16; 25, 16)	(5, 16; 1, 7)	(7, 13; 1, 9)					

$a = 6$ :

(6, 1; 24, 7)	(4, 15; 29, 12)	(0, 10; 27, 13)	(14, 17; 22, 25)	(12, 6; 3, 19)	(16, 4; 6, 21)	(15, 0; 14, 8)	(3, 14; 28, 8)	(3, 10; 12, 3)
(16, 10; 24, 16)	(5, 2; 19, 26)	(10, 12; 5, 26)	(0, 6; 5, 10)	(9, 6; 4, 8)	(15, 11; 4, 10)	(7, 11; 1, 8)	(1, 17; 15, 26)	(10, 5; 15, 2)
(7, 16; 4, 26)	(1, 10; 11, 28)	(13, 15; 16, 6)	(16, 14; 19, 15)	(6, 4; 1, 28)	(13, 6; 11, 21)	(8, 1; 6, 8)	(0, 16; 11, 22)	(17, 2; 8, 24)
(2, 7; 25, 20)	(3, 4; 19, 23)	(10, 9; 29, 6)	(1, 5; 4, 16)	(14, 11; 3, 5)	(5, 9; 22, 1)	(2, 13; 1, 29)	(6, 14; 18, 23)	(9, 1; 5, 20)
(12, 9; 9, 15)	(7, 10; 7, 18)	(4, 14; 24, 13)	(12, 2; 21, 13)	(12, 16; 2, 8)	(8, 12; 4, 27)	(14, 0; 16, 7)	(5, 15; 18, 28)	(5, 14; 12, 20)
(13, 11; 2, 19)	(13, 16; 5, 0)	(6, 8; 9, 0)	(13, 0; 15, 4)	(3, 5; 24, 0)	(1, 13; 12, 22)	(12, 0; 28, 23)	(14, 10; 4, 9)	(1, 14; 29, 14)
(13, 10; 17, 8)	(2, 9; 27, 23)	(1, 4; 27, 25)	(12, 15; 7, 22)	(12, 11; 18, 29)	(11, 16; 14, 25)	(4, 17; 4, 3)	(3, 9; 13, 26)	(7, 4; 5, 22)
(15, 10; 0, 25)	(13, 7; 23, 9)	(2, 11; 28, 0)	(15, 1; 19, 21)	(12, 5; 14, 6)	(12, 17; 12, 11)	(0, 3; 9, 21)	(5, 11; 17, 13)	(7, 17; 27, 0)
(8, 13; 10, 28)	(16, 2; 12, 10)	(15, 6; 2, 20)	(3, 12; 20, 10)	(16, 15; 3, 9)	(8, 14; 21, 2)	(9, 17; 2, 7)	(9, 15; 24, 11)	(11, 8; 24, 12)
(13, 4; 18, 26)	(8, 16; 7, 20)	(6, 2; 22, 16)	(8, 15; 5, 13)	(3, 7; 16, 2)	(12, 14; 17, 1)	(4, 0; 20, 17)	(4, 11; 15, 7)	(15, 2; 15, 17)
(5, 7; 29, 3)	(7, 9; 21, 28)	(14, 9; 10, 0)	(17, 15; 1, 23)	(1, 2; 9, 18)	(7, 1; 10, 17)	(0, 7; 24, 19)	(10, 17; 10, 19)	(0, 9; 25, 12)
(16, 1; 13, 23)	(11, 3; 27, 11)	(16, 6; 17, 27)	(13, 5; 27, 7)	(17, 5; 21, 5)	(17, 13; 20, 13)	(2, 4; 14, 2)	(14, 7; 11, 6)	(11, 17; 9, 16)
(2, 8; 3, 11)	(4, 12; 0, 16)	(9, 13; 3, 14)	(0, 8; 26, 29)	(6, 3; 29, 25)	(3, 17; 14, 17)	(0, 17; 6, 18)	(6, 11; 6, 26)	(3, 8; 15, 22)
(3, 16; 1, 18)	(5, 8; 23, 25)	(8, 10; 1, 14)						

$a = 7$ :

(10, 17; 13, 28)	(5, 7; 5, 17)	(2, 8; 29, 10)	(15, 6; 9, 27)	(1, 3; 24, 20)	(7, 9; 9, 31)	(6, 10; 25, 18)	(10, 2; 30, 17)	(14, 2; 25, 16)
(6, 11; 1, 31)	(13, 2; 9, 15)	(11, 16; 12, 4)	(17, 1; 29, 26)	(1, 9; 25, 30)	(9, 11; 3, 28)	(12, 17; 7, 7)	(8, 15; 1, 24)	(12, 1; 6, 16)
(8, 4; 13, 30)	(7, 12; 3, 29)	(3, 9; 21, 12)	(2, 7; 0, 26)	(16, 6; 21, 3)	(11, 13; 5, 8)	(2, 1; 11, 14)	(15, 5; 16, 23)	(6, 8; 28, 5)
(12, 4; 5, 0)	(2, 17; 1, 23)	(9, 2; 24, 13)	(6, 2; 2, 8)	(14, 13; 2, 12)	(9, 14; 0, 11)	(14, 6; 30, 7)	(17, 14; 24, 19)	(7, 3; 30, 27)
(3, 4; 1, 15)	(13, 15; 4, 20)	(15, 2; 22, 3)	(10, 13; 19, 29)	(10, 3; 3, 8)	(12, 3; 22, 28)	(9, 6; 26, 22)	(14, 4; 31, 23)	(14, 7; 22, 6)
(3, 8; 2, 23)	(4, 6; 6, 20)	(4, 0; 22, 19)	(15, 17; 10, 2)	(15, 1; 5, 18)	(8, 14; 4, 27)	(3, 16; 29, 17)	(9, 16; 10, 6)	(1, 6; 23, 17)
(1, 4; 7, 27)	(3, 5; 13, 25)	(11, 12; 30, 15)	(10, 1; 12, 10)	(17, 9; 5, 27)	(8, 10; 9, 6)	(7, 15; 11, 21)	(16, 4; 24, 18)	(12, 6; 10, 19)
(15, 10; 14, 0)	(0, 5; 21, 27)	(5, 2; 31, 12)	(15, 16; 7, 13)	(8, 16; 0, 25)	(14, 3; 14, 10)	(11, 2; 19, 27)	(16, 14; 5, 20)	(5, 13; 30, 1)
(13, 17; 0, 22)	(0, 13; 17, 6)	(0, 15; 30, 12)	(7, 4; 4, 25)	(7, 1; 8, 19)	(11, 7; 2, 24)	(3, 15; 19, 31)	(8, 5; 14, 22)	(16, 10; 27, 11)
(17, 5; 6, 15)	(9, 10; 1, 7)	(15, 11; 25, 6)	(12, 0; 11, 13)	(4, 10; 2, 16)	(6, 17; 4, 11)	(6, 3; 0, 16)	(3, 13; 11, 18)	(0, 9; 4, 29)
(14, 12; 1, 8)	(0, 11; 16, 10)	(11, 8; 26, 11)	(4, 13; 28, 21)	(5, 11; 7, 0)	(9, 5; 2, 20)	(12, 9; 14, 23)	(11, 17; 9, 18)	(0, 17; 25, 14)
(0, 3; 26, 9)	(4, 17; 3, 12)	(16, 5; 26, 9)	(16, 12; 2, 9)	(11, 4; 14, 29)	(13, 16; 14, 16)	(10, 12; 4, 31)	(14, 1; 9, 21)	(12, 2; 18, 21)
(5, 1; 4, 28)	(4, 15; 26, 17)	(8, 0; 31, 15)	(17, 7; 16, 20)	(8, 12; 12, 20)	(14, 10; 15, 26)	(16, 1; 15, 22)	(1, 13; 31, 13)	(7, 0; 7, 18)
(6, 5; 29, 24)	(7, 13; 23, 10)	(13, 8; 3, 7)	(8, 17; 8, 21)	(14, 11; 13, 17)	(9, 15; 8, 15)	(0, 10; 24, 5)	(14, 5; 3, 18)	(16, 7; 1, 28)
(0, 2; 20, 28)	(0, 16; 8, 23)							

$a = 8$ :

(7, 9; 28, 6)	(0, 16; 10, 21)	(2, 7; 31, 10)	(17, 1; 21, 6)	(11, 0; 4, 18)	(1, 2; 19, 23)	(13, 0; 19, 6)	(4, 16; 24, 1)	(3, 13; 16, 30)
(16, 1; 27, 22)	(0, 6; 10, 30)	(4, 0; 17, 30)	(4, 3; 13, 23)	(17, 3; 10, 18)	(3, 16; 3, 20)	(6, 3; 25, 28)	(17, 13; 4, 8)	(1, 14; 10, 7)
(8, 3; 22, 24)	(5, 7; 30, 4)	(1, 7; 16, 26)	(13, 8; 29, 21)	(14, 8; 20, 11)	(15, 8; 14, 0)	(7, 10; 1, 8)	(9, 10; 5, 26)	(12, 14; 30, 6)
(15, 11; 26, 10)	(8, 12; 10, 15)	(15, 10; 13, 16)	(8, 7; 23, 3)	(16, 2; 8, 16)	(5, 17; 7, 27)	(10, 16; 15, 6)	(14, 17; 22, 17)	(4, 8; 12, 6)
(2, 13; 9, 20)	(9, 1; 13, 20)	(3, 9; 0, 15)	(5, 13; 13, 1)	(13, 16; 23, 14)	(12, 10; 28, 9)	(12, 2; 22, 13)	(2, 14; 27, 3)	(4, 7; 5, 22)
(5, 2; 2, 29)	(9, 15; 21, 27)	(1, 4; 31, 29)	(17, 7; 24, 2)	(11, 7; 29, 9)	(1, 11; 11, 17)	(2, 17; 12, 0)	(17, 8; 1, 5)	(4, 12; 16, 0)
(15, 12; 4, 3)	(0, 9; 14, 7)	(8, 6; 27, 31)	(12, 3; 11, 29)	(0, 15; 15, 24)	(8, 10; 7, 2)	(3, 14; 26, 1)	(13, 15; 5, 31)	(9, 11; 3, 30)
(6, 9; 2, 9)	(17, 6; 29, 16)	(10, 4; 18, 14)	(12, 5; 23, 18)	(10, 14; 12, 4)	(16, 5; 0, 28)	(6, 12; 8, 19)	(2, 11; 1, 14)	(5, 10; 31, 3)
(3, 15; 2, 12)	(10, 3; 19, 27)	(9, 17; 25, 11)	(13, 9; 22, 10)	(13, 14; 2, 15)	(0, 7; 27, 20)	(1, 13; 18, 12)	(16, 14; 5, 13)	(13, 10; 0, 11)
(6, 1; 24, 5)	(5, 9; 12, 24)	(8, 16; 26, 9)	(2, 6; 26, 21)	(8, 2; 25, 30)	(7, 12; 21, 7)	(0, 3; 31, 8)	(0, 6; 11, 23)	(6, 16; 7, 18)
(0, 17; 13, 28)	(11, 14; 31, 24)	(14, 9; 8, 23)	(14, 5; 19, 14)	(8, 11; 13, 8)	(9, 16; 29, 4)	(5, 1; 15, 25)	(17, 15; 23, 9)	(14, 4; 21, 25)
(12, 17; 14, 20)	(12, 0; 5, 12)	(4, 6; 4, 20)	(1, 8; 28, 4)	(14, 7; 0, 18)	(0, 10; 25, 29)	(6, 13; 3, 17)	(9, 12; 1, 31)	(11, 6; 6, 0)
(16, 12; 17, 2)	(17, 11; 19, 15)	(11, 5; 5, 16)	(5, 3; 17, 21)	(1, 15; 30, 8)	(15, 6; 1, 22)	(3, 1; 9, 14)	(7, 16; 11, 19)	(15, 5; 6, 20)
(0, 5; 26, 22)	(0, 14; 16, 9)	(11, 16; 12, 25)	(2, 10; 17, 24)	(7, 15; 25, 17)	(15, 4; 7, 19)	(17, 4; 26, 3)	(2, 15; 11, 18)	(11, 4; 27, 2)
(2, 4; 15, 28)	(11, 13; 7, 28)							

$a = 9$ :

(17, 2; 9, 0)	(3, 0; 10, 12)	(6, 10; 2, 4)	(15, 6; 34, 3)	(4, 9; 34, 20)	(8, 12; 13, 2)	(3, 6; 33, 23)	(7, 0; 30, 6)	(16, 10; 3, 29)
(3, 5; 26, 19)	(8, 1; 26, 33)	(11, 17; 11, 30)	(11, 4; 29, 33)	(16, 7; 19, 28)	(1, 2; 35, 12)	(12, 15; 19, 32)	(16, 3; 24, 16)	(0, 10; 14, 26)
(1, 7; 31, 25)	(15, 8; 0, 10)	(9, 6; 26, 30)	(0, 2; 34, 11)	(11, 0; 25, 16)	(5, 12; 18, 5)	(15, 5; 24, 6)	(15, 4; 25, 15)	(12, 1; 9, 23)
(13, 1; 30, 21)	(13, 15; 16, 12)	(2, 9; 29, 10)	(4, 3; 30, 2)	(16, 5; 30, 14)	(17, 3; 20, 25)	(4, 6; 35, 16)	(1, 6; 24, 29)	(6, 5; 22, 32)
(12, 2; 20, 1)	(8, 16; 12, 5)	(11, 12; 4, 8)	(14, 3; 11, 21)	(6, 11; 31, 0)	(13, 16; 31, 11)	(2, 5; 31, 27)	(4, 13; 0, 18)	(17, 0; 5, 17)
(13, 17; 19, 2)	(0, 4; 24, 28)	(5, 11; 3, 12)	(0, 5; 4, 29)	(5, 9; 1, 23)	(0, 16; 23, 18)	(15, 2; 21, 26)	(4, 1; 5, 32)	(9, 17; 12, 24)
(11, 14; 1, 15)	(13, 10; 35, 17)	(6, 0; 19, 27)	(9, 12; 33, 28)	(9, 10; 27, 11)	(11, 15; 17, 27)	(10, 7; 5, 24)	(6, 12; 7, 17)	(8, 0; 15, 20)
(3, 1; 28, 17)	(1, 17; 16, 27)	(15, 10; 13, 8)	(1, 10; 34, 18)	(17, 10; 6, 10)	(17, 15; 23, 4)	(7, 15; 20, 18)	(17, 12; 3, 31)	(13, 9; 4, 14)
(3, 12; 0, 29)	(15, 3; 35, 1)	(14, 12; 14, 34)	(9, 14; 35, 8)	(9, 3; 3, 13)	(13, 8; 7, 29)	(2, 4; 14, 3)	(14, 5; 16, 2)	(11, 9; 32, 6)
(8, 5; 35, 25)	(7, 5; 17, 34)	(11, 1; 10, 13)	(1, 15; 14, 11)	(4, 14; 17, 12)	(13, 11; 5, 28)	(7, 3; 32, 9)	(7, 3; 32, 9)	(7, 17; 29, 22)
(9, 1; 15, 22)	(8, 3; 8, 27)	(5, 17; 15, 21)	(6, 14; 5, 10)	(16, 9; 25, 0)	(2, 13; 15, 23)	(4, 12; 22, 6)	(2, 16; 13, 17)	(3, 13; 34, 22)
(6, 17; 18, 8)	(10, 3; 15, 31)	(7, 14; 0, 33)	(7, 8; 3, 4)	(16, 14; 27, 20)	(6, 2; 25, 28)	(10, 12; 12, 30)	(16, 15; 9, 22)	(10, 2; 16, 33)
(16, 11; 2, 7)	(4, 17; 13, 1)	(12, 0; 21, 35)	(14, 13; 3, 6)	(4, 8; 31, 21)	(17, 8; 28, 14)	(1, 14; 4, 19)	(2, 7; 8, 2)	(12, 7; 16, 11)
(6, 13; 9, 20)	(11, 3; 14, 18)	(12, 16; 10, 15)	(2, 8; 30, 22)	(10, 14; 25, 9)	(16, 1; 8, 6)	(5, 10; 28, 0)	(5, 1; 20, 7)	(6, 8; 6, 11)
(7, 4; 23, 27)	(14, 0; 13, 22)	(4, 10; 19, 7)	(0, 9; 9, 31)	(4, 16; 26, 4)	(8, 14; 23, 24)	(2, 11; 24, 19)	(7, 11; 26, 35)	(9, 15; 2, 5)
(14, 2; 32, 18)	(14, 17; 7, 26)	(9, 7; 7, 21)	(13, 7; 10, 1)	(11, 8; 9, 34)	(16, 6; 21, 1)	(13, 0; 32, 8)	(0, 15; 7, 33)	(5, 13; 13, 33)

C.  $HSAS(s, v; 3, 3)$  and  $HSAS(s, v; 5, 3)$  with  $v \in \{11, 15, 19\}$

*Lemma 8.13:* There exists an  $HSAS(s, 11; 3, 3)$  for each  $s \in \{11, 13, 15, 17, 19\}$ .

*Proof:* Let  $V = I_{11}$  and  $S = I_s$ . Let  $W = \{8, 9, 10\}$  and  $T = \{s - 3, s - 2, s - 1\}$ . The desired HSASs filled with pairs of points from  $V$  and indexed by  $S$  are presented as follows.

$s = 11$ :

(3, 5; 2, 10)	(6, 2; 4, 3)	(4, 8; 1, 2)	(8, 7; 6, 5)	(2, 9; 1, 0)	(6, 3; 0, 8)	(4, 7; 8, 3)	(3, 7; 9, 1)	(5, 10; 3, 7)	(2, 1; 6, 9)
(8, 3; 4, 7)	(5, 6; 5, 9)	(0, 6; 6, 10)	(3, 9; 5, 3)	(1, 8; 3, 0)	(4, 10; 6, 0)	(0, 2; 8, 2)	(1, 10; 2, 4)	(4, 2; 5, 10)	(0, 4; 7, 9)
(1, 7; 7, 10)	(10, 0; 1, 5)	(6, 9; 7, 2)	(5, 1; 8, 1)	(0, 7; 0, 4)	(5, 9; 4, 6)				

$s = 13$ :

(10, 5; 2, 3)	(9, 0; 9, 0)	(10, 4; 5, 7)	(2, 4; 4, 12)	(0, 6; 7, 3)	(0, 7; 12, 2)	(7, 6; 9, 5)	(3, 10; 8, 4)	(8, 1; 1, 4)	(2, 1; 2, 11)
(5, 1; 9, 7)	(9, 3; 1, 7)	(1, 6; 12, 8)	(5, 0; 4, 10)	(10, 1; 6, 0)	(4, 0; 6, 11)	(7, 3; 0, 11)	(9, 2; 5, 3)	(8, 3; 9, 3)	(8, 6; 6, 2)
(3, 5; 12, 6)	(6, 5; 11, 1)	(4, 7; 1, 3)	(2, 5; 8, 0)	(1, 3; 10, 5)	(9, 4; 8, 2)	(4, 6; 0, 10)	(10, 2; 9, 1)	(9, 7; 6, 4)	(8, 0; 5, 8)
(2, 7; 7, 10)									

$s = 15$ :

(7, 4; 8, 7)	(5, 6; 4, 9)	(3, 4; 3, 5)	(4, 5; 0, 14)	(7, 0; 14, 10)	(4, 6; 12, 11)	(0, 3; 11, 1)	(2, 8; 6, 1)	(1, 8; 11, 5)	(9, 7; 6, 2)
(0, 9; 8, 9)	(8, 7; 0, 9)	(8, 3; 7, 10)	(2, 7; 5, 12)	(9, 2; 7, 11)	(3, 9; 4, 0)	(0, 6; 6, 0)	(4, 10; 9, 1)	(10, 6; 7, 5)	(3, 6; 14, 8)
(5, 8; 3, 8)	(1, 3; 12, 6)	(9, 6; 10, 3)	(1, 10; 8, 4)	(10, 2; 2, 0)	(5, 3; 13, 2)	(4, 8; 4, 2)	(7, 10; 3, 11)	(5, 9; 1, 5)	(2, 1; 9, 14)
(10, 5; 6, 10)	(1, 4; 13, 10)	(6, 7; 13, 1)	(0, 1; 3, 2)	(0, 2; 4, 13)	(0, 5; 7, 12)				

$s = 17$ :

(2, 10; 1, 7)	(2, 7; 16, 9)	(2, 6; 14, 11)	(5, 3; 3, 16)	(2, 5; 10, 0)	(1, 3; 4, 5)	(0, 5; 12, 2)	(10, 1; 13, 2)	(4, 2; 4, 6)	(8, 6; 10, 4)
(10, 0; 8, 5)	(5, 4; 14, 5)	(7, 6; 12, 0)	(8, 3; 2, 11)	(8, 2; 5, 3)	(10, 7; 11, 10)	(4, 0; 11, 0)	(9, 1; 11, 12)	(10, 6; 3, 6)	(4, 6; 16, 2)
(7, 3; 15, 6)	(5, 10; 4, 9)	(4, 8; 12, 7)	(7, 9; 5, 2)	(4, 9; 8, 10)	(6, 1; 8, 1)	(8, 1; 0, 6)	(6, 9; 13, 9)	(9, 5; 1, 6)	(0, 8; 1, 9)
(0, 2; 15, 13)	(1, 0; 7, 16)	(4, 1; 9, 15)	(8, 7; 13, 8)	(4, 3; 13, 1)	(3, 9; 0, 7)	(0, 3; 10, 14)	(5, 6; 15, 7)	(3, 2; 12, 8)	(7, 1; 3, 14)
(0, 9; 3, 4)									

$s = 19$ :

(6, 5; 13, 2)	(2, 9; 6, 1)	(6, 3; 1, 16)	(3, 0; 7, 5)	(5, 9; 12, 3)	(4, 6; 12, 18)	(0, 9; 0, 15)	(6, 7; 17, 4)	(6, 9; 7, 10)	(7, 2; 12, 14)
(1, 0; 17, 1)	(8, 0; 13, 12)	(4, 9; 11, 5)	(0, 6; 3, 14)	(10, 6; 0, 6)	(4, 2; 0, 13)	(4, 5; 16, 15)	(3, 2; 18, 8)	(5, 8; 0, 11)	(9, 1; 8, 13)
(1, 4; 4, 6)	(4, 0; 10, 2)	(5, 2; 17, 5)	(0, 7; 11, 18)	(8, 3; 10, 6)	(1, 5; 18, 10)	(8, 1; 14, 2)	(3, 4; 17, 14)	(4, 10; 9, 7)	(9, 7; 9, 2)
(1, 10; 12, 5)	(2, 8; 15, 4)	(6, 1; 15, 11)	(3, 10; 2, 15)	(5, 3; 9, 4)	(2, 10; 11, 10)	(2, 1; 3, 7)	(5, 10; 1, 14)	(1, 7; 16, 0)	(10, 7; 13, 3)
(0, 2; 16, 9)	(8, 6; 9, 5)	(10, 0; 4, 8)	(8, 4; 3, 8)	(7, 8; 1, 7)	(5, 7; 6, 8)				

*Lemma 8.14:* There exists an  $HSAS(s, 15; 3, 3)$  for each  $s \in \{15, 17, \dots, 27\}$ .

*Proof:* Let  $V = I_{15}$  and  $S = I_s$ . Let  $W = \{12, 13, 14\}$  and  $T = \{s - 3, s - 2, s - 1\}$ . The desired HSASs filled with pairs of points from  $V$  and indexed by  $S$  are presented as follows.

$s = 15$ :

(12, 9; 1, 0)	(7, 8; 5, 13)	(10, 12; 6, 9)	(6, 14; 3, 9)	(5, 1; 10, 1)	(4, 13; 6, 10)	(11, 2; 14, 1)	(11, 9; 13, 6)	(9, 6; 11, 5)	(0, 5; 0, 9)
(0, 8; 8, 14)	(2, 0; 13, 3)	(10, 1; 0, 13)	(11, 4; 11, 2)	(2, 1; 6, 12)	(10, 4; 14, 3)	(12, 2; 7, 2)	(3, 5; 12, 11)	(4, 5; 8, 13)	(7, 5; 7, 4)
(8, 6; 0, 6)	(2, 14; 0, 10)	(1, 8; 3, 2)	(0, 10; 11, 7)	(11, 12; 4, 8)	(12, 8; 10, 11)	(7, 1; 9, 14)	(2, 3; 9, 8)	(10, 7; 2, 12)	(12, 5; 5, 3)
(14, 1; 4, 11)	(11, 6; 7, 12)	(9, 13; 3, 4)	(2, 4; 4, 5)	(8, 10; 1, 4)	(13, 7; 11, 0)	(0, 14; 6, 5)	(4, 8; 12, 9)	(11, 13; 5, 9)	(9, 3; 14, 7)
(14, 9; 8, 2)	(6, 3; 13, 4)	(0, 13; 2, 1)	(13, 1; 8, 7)	(3, 11; 0, 3)	(0, 9; 10, 12)	(3, 7; 1, 6)	(6, 5; 2, 14)	(7, 6; 8, 10)	(3, 10; 5, 10)
(4, 14; 1, 7)									

$s = 17$ :

(12, 10; 1, 0)	(3, 5; 14, 1)	(11, 13; 10, 0)	(7, 0; 1, 5)	(0, 6; 7, 6)	(13, 8; 1, 4)	(9, 5; 9, 15)	(2, 10; 5, 14)	(5, 4; 12, 16)	(8, 14; 10, 13)
(3, 2; 2, 12)	(11, 0; 8, 16)	(10, 7; 11, 16)	(1, 11; 15, 7)	(4, 3; 11, 15)	(5, 1; 0, 3)	(4, 13; 5, 9)	(2, 8; 16, 7)	(5, 12; 5, 10)	(9, 0; 4, 12)
(11, 9; 1, 11)	(2, 6; 0, 15)	(3, 13; 7, 8)	(5, 7; 4, 6)	(9, 2; 10, 8)	(11, 8; 14, 6)	(10, 5; 8, 2)	(10, 8; 3, 12)	(1, 3; 13, 9)	(6, 1; 2, 16)
(7, 14; 12, 8)	(2, 11; 4, 9)	(8, 1; 8, 11)	(3, 8; 0, 5)	(12, 11; 12, 13)	(13, 1; 12, 6)	(12, 4; 2, 6)	(13, 9; 2, 13)	(10, 6; 9, 10)	(2, 14; 1, 6)
(5, 14; 11, 7)	(12, 6; 4, 11)	(6, 7; 13, 14)	(13, 0; 3, 11)	(9, 4; 0, 14)	(9, 3; 16, 6)	(7, 8; 15, 2)	(1, 14; 5, 4)	(7, 3; 10, 3)	(12, 7; 7, 9)
(1, 0; 10, 14)	(10, 4; 4, 7)	(4, 6; 8, 1)	(6, 9; 5, 3)	(4, 2; 3, 13)	(0, 14; 9, 0)	(0, 10; 13, 15)	(11, 14; 2, 3)		

$s = 19$ :

(7, 12; 7, 6)	(2, 12; 8, 2)	(0, 9; 4, 2)	(10, 2; 12, 15)	(3, 5; 10, 6)	(6, 13; 10, 4)	(5, 12; 1, 9)	(7, 13; 8, 12)	(7, 14; 15, 4)
(6, 0; 16, 0)	(10, 12; 14, 4)	(11, 7; 0, 2)	(5, 8; 16, 8)	(13, 10; 13, 2)	(7, 0; 9, 14)	(14, 9; 0, 12)	(4, 13; 0, 1)	(10, 3; 1, 5)
(8, 14; 10, 14)	(9, 10; 8, 9)	(11, 6; 18, 7)	(3, 4; 3, 18)	(3, 12; 13, 11)	(11, 3; 9, 12)	(3, 13; 14, 7)	(4, 8; 2, 11)	(1, 0; 11, 6)
(0, 11; 15, 1)	(10, 6; 17, 11)	(4, 1; 17, 14)	(5, 1; 7, 4)	(6, 5; 2, 12)	(2, 8; 4, 6)	(10, 7; 18, 10)	(11, 9; 17, 10)	(4, 12; 10, 12)
(13, 5; 5, 11)	(8, 7; 13, 1)	(1, 3; 16, 15)	(9, 6; 14, 15)	(0, 10; 3, 7)	(14, 11; 3, 13)	(8, 6; 5, 9)	(1, 6; 8, 3)	(9, 7; 11, 16)
(14, 6; 1, 6)	(9, 1; 18, 13)	(8, 3; 0, 17)	(0, 14; 8, 5)	(4, 2; 16, 9)	(5, 4; 13, 15)	(1, 2; 10, 1)	(5, 7; 17, 3)	(12, 1; 5, 0)
(8, 12; 15, 3)	(0, 8; 18, 12)	(1, 14; 9, 2)	(4, 11; 8, 4)	(10, 11; 16, 6)	(9, 13; 3, 6)	(5, 2; 0, 18)	(2, 0; 17, 13)	(4, 9; 5, 7)
(2, 11; 14, 5)	(2, 14; 7, 11)							

$s = 21$ :

(8, 14; 13, 6)	(12, 6; 15, 1)	(0, 7; 17, 20)	(6, 11; 8, 13)	(7, 11; 4, 18)	(7, 1; 14, 10)	(0, 13; 8, 0)	(8, 7; 19, 2)	(5, 2; 18, 15)
(6, 8; 18, 10)	(8, 0; 15, 4)	(12, 5; 12, 6)	(13, 10; 10, 16)	(4, 1; 4, 20)	(3, 11; 5, 15)	(6, 3; 9, 19)	(1, 13; 15, 6)	(7, 2; 16, 11)
(4, 9; 8, 17)	(9, 8; 9, 16)	(4, 11; 19, 3)	(3, 14; 8, 12)	(6, 2; 6, 4)	(3, 10; 3, 14)	(5, 8; 8, 3)	(4, 6; 2, 5)	(4, 5; 1, 0)
(8, 11; 12, 17)	(11, 12; 14, 16)	(10, 4; 6, 18)	(0, 4; 12, 13)	(0, 10; 1, 19)	(1, 6; 16, 12)	(13, 11; 1, 7)	(9, 5; 4, 5)	(9, 11; 11, 6)
(2, 14; 17, 3)	(1, 10; 8, 7)	(3, 13; 17, 4)	(12, 0; 5, 11)	(11, 10; 9, 20)	(7, 14; 7, 5)	(2, 11; 0, 10)	(8, 13; 14, 5)	(4, 2; 9, 14)
(0, 3; 2, 6)	(4, 14; 16, 15)	(9, 6; 20, 7)	(2, 10; 12, 5)	(0, 1; 9, 18)	(9, 2; 19, 13)	(5, 14; 10, 9)	(7, 12; 3, 0)	(10, 14; 11, 4)
(14, 1; 0, 2)	(13, 5; 2, 13)	(2, 8; 20, 1)	(0, 5; 14, 7)	(5, 1; 19, 11)	(12, 2; 2, 8)	(3, 9; 0, 18)	(14, 9; 14, 1)	(6, 13; 3, 11)
(8, 3; 7, 11)	(3, 5; 20, 16)	(9, 10; 2, 15)	(7, 13; 9, 12)	(9, 0; 10, 3)	(6, 10; 17, 0)	(12, 4; 7, 10)	(1, 12; 13, 17)	(3, 7; 1, 13)



$s = 23$ :

(9, 4; 18, 4)	(13, 6; 8, 12)	(4, 1; 15, 12)	(7, 6; 15, 13)	(10, 7; 17, 11)	(10, 2; 15, 20)	(14, 9; 19, 0)	(0, 2; 21, 17)	(13, 9; 2, 9)
(9, 1; 6, 8)	(11, 12; 5, 8)	(7, 13; 0, 14)	(12, 7; 6, 12)	(8, 0; 8, 20)	(6, 3; 6, 21)	(14, 0; 5, 16)	(7, 5; 4, 1)	(1, 7; 20, 7)
(3, 0; 2, 19)	(8, 14; 9, 17)	(14, 5; 11, 6)	(8, 5; 2, 18)	(10, 4; 5, 22)	(13, 11; 13, 16)	(14, 2; 12, 18)	(1, 12; 0, 3)	(3, 8; 13, 7)
(3, 14; 4, 8)	(7, 9; 3, 21)	(12, 9; 17, 13)	(4, 3; 3, 20)	(4, 11; 21, 19)	(2, 1; 14, 13)	(8, 13; 11, 3)	(5, 13; 5, 17)	(6, 5; 20, 19)
(6, 1; 16, 17)	(0, 13; 18, 15)	(5, 3; 10, 14)	(2, 7; 19, 16)	(7, 8; 10, 5)	(6, 8; 0, 1)	(10, 1; 19, 18)	(10, 11; 14, 2)	(5, 12; 16, 15)
(4, 8; 16, 6)	(11, 5; 12, 9)	(11, 1; 1, 22)	(0, 9; 12, 14)	(14, 4; 14, 1)	(9, 6; 11, 5)	(0, 10; 6, 1)	(12, 6; 18, 10)	(12, 4; 7, 2)
(6, 2; 3, 2)	(1, 14; 10, 2)	(3, 10; 12, 16)	(4, 7; 8, 9)	(2, 9; 7, 1)	(9, 11; 20, 10)	(5, 2; 8, 22)	(6, 0; 9, 22)	(9, 8; 22, 15)
(4, 2; 11, 0)	(1, 3; 5, 9)	(5, 0; 3, 7)	(6, 10; 7, 4)	(0, 4; 10, 13)	(3, 11; 17, 0)	(11, 14; 15, 7)	(10, 5; 0, 21)	(8, 12; 19, 14)
(3, 7; 18, 22)	(12, 3; 11, 1)	(1, 8; 21, 4)	(10, 14; 3, 13)	(11, 0; 4, 11)	(2, 13; 6, 10)	(2, 12; 4, 9)		

$s = 25$ :

(3, 7; 3, 17)	(9, 8; 11, 1)	(5, 10; 8, 1)	(9, 1; 16, 22)	(8, 11; 21, 24)	(4, 3; 5, 24)	(4, 5; 17, 23)	(8, 14; 5, 8)	(12, 0; 1, 20)
(14, 10; 6, 10)	(0, 13; 13, 4)	(14, 0; 21, 19)	(11, 4; 11, 10)	(4, 10; 19, 20)	(1, 7; 1, 24)	(8, 10; 2, 7)	(12, 10; 13, 0)	(13, 4; 2, 8)
(0, 8; 22, 15)	(5, 8; 16, 14)	(11, 13; 9, 1)	(12, 5; 6, 12)	(2, 14; 3, 13)	(2, 13; 16, 0)	(6, 13; 10, 7)	(6, 11; 13, 23)	(7, 11; 7, 22)
(8, 12; 19, 3)	(1, 8; 9, 13)	(3, 10; 22, 9)	(2, 4; 22, 1)	(9, 0; 17, 8)	(7, 10; 14, 15)	(9, 3; 7, 18)	(6, 0; 24, 11)	(14, 6; 2, 18)
(7, 0; 2, 10)	(2, 8; 20, 4)	(1, 13; 15, 11)	(7, 14; 0, 4)	(1, 14; 17, 20)	(12, 11; 5, 17)	(5, 3; 15, 4)	(6, 5; 22, 21)	(11, 14; 14, 12)
(11, 0; 16, 18)	(12, 7; 11, 9)	(9, 2; 12, 21)	(6, 10; 16, 12)	(9, 4; 14, 0)	(14, 4; 15, 9)	(10, 13; 17, 21)	(3, 14; 1, 16)	(7, 13; 12, 19)
(3, 13; 14, 20)	(2, 6; 15, 8)	(4, 0; 12, 3)	(2, 5; 5, 10)	(6, 8; 17, 0)	(1, 4; 7, 21)	(2, 1; 14, 2)	(6, 7; 5, 20)	(7, 4; 13, 16)
(2, 10; 18, 24)	(6, 4; 18, 6)	(0, 2; 9, 7)	(7, 2; 6, 23)	(1, 11; 3, 4)	(11, 3; 6, 8)	(12, 6; 14, 4)	(9, 12; 10, 15)	(1, 6; 19, 6)
(9, 13; 6, 5)	(10, 9; 23, 4)	(5, 13; 18, 3)	(1, 0; 23, 5)	(8, 3; 12, 23)	(5, 9; 24, 13)	(9, 6; 9, 3)	(12, 1; 18, 8)	(5, 14; 11, 7)
(2, 3; 19, 11)	(3, 12; 2, 21)	(11, 9; 2, 19)	(1, 3; 0, 10)	(5, 11; 0, 20)				

$s = 27$ :

(3, 12; 0, 23)	(1, 7; 0, 22)	(6, 13; 6, 17)	(7, 2; 17, 23)	(11, 8; 0, 11)	(1, 11; 24, 10)	(14, 2; 3, 11)	(3, 14; 12, 15)	(6, 8; 12, 14)
(14, 8; 9, 20)	(8, 5; 18, 26)	(13, 1; 2, 11)	(6, 7; 24, 1)	(1, 14; 13, 18)	(6, 0; 8, 22)	(7, 9; 16, 5)	(12, 5; 11, 12)	(5, 2; 21, 15)
(5, 3; 2, 8)	(8, 1; 16, 3)	(13, 10; 23, 5)	(0, 8; 23, 4)	(3, 1; 7, 26)	(5, 1; 9, 1)	(5, 7; 6, 13)	(14, 11; 16, 1)	(2, 13; 0, 10)
(3, 7; 4, 11)	(10, 5; 10, 16)	(7, 14; 8, 14)	(11, 6; 15, 23)	(12, 6; 18, 16)	(7, 10; 26, 15)	(6, 9; 26, 11)	(8, 13; 21, 1)	(1, 4; 5, 14)
(5, 13; 4, 22)	(14, 9; 10, 2)	(0, 9; 15, 24)	(9, 12; 6, 14)	(10, 1; 12, 19)	(12, 4; 13, 10)	(10, 9; 17, 1)	(3, 2; 14, 1)	(5, 6; 25, 3)
(9, 1; 23, 8)	(9, 8; 22, 19)	(13, 3; 20, 19)	(11, 4; 18, 2)	(4, 14; 22, 23)	(8, 7; 7, 25)	(8, 10; 2, 13)	(0, 3; 13, 25)	(8, 12; 15, 8)
(11, 7; 19, 21)	(1, 2; 20, 25)	(10, 3; 21, 24)	(12, 11; 3, 20)	(9, 4; 21, 4)	(0, 5; 0, 14)	(4, 5; 24, 17)	(4, 13; 7, 15)	(11, 0; 5, 7)
(0, 1; 21, 6)	(0, 2; 2, 16)	(2, 12; 19, 5)	(10, 0; 9, 3)	(8, 2; 6, 24)	(1, 12; 4, 17)	(10, 2; 18, 22)	(13, 0; 18, 12)	(0, 7; 10, 20)
(10, 6; 4, 20)	(6, 2; 7, 13)	(4, 3; 16, 6)	(4, 0; 26, 1)	(3, 8; 5, 10)	(9, 5; 20, 7)	(7, 12; 9, 2)	(10, 11; 8, 6)	(6, 14; 5, 21)
(4, 6; 19, 0)	(13, 11; 9, 14)	(9, 3; 18, 9)	(10, 4; 11, 25)	(14, 0; 17, 19)	(3, 11; 22, 17)	(2, 11; 4, 26)	(9, 13; 3, 13)	(9, 11; 25, 12)
(4, 2; 8, 9)	(4, 7; 3, 12)	(10, 14; 0, 7)						

**Lemma 8.15:** There exists an HSAS( $s, 19; 3, 3$ ) for each  $s \in \{19, 21, \dots, 35\}$ .

*Proof:* Let  $V = I_{19}$  and  $S = I_s$ . Let  $W = \{16, 17, 18\}$  and  $T = \{s - 3, s - 2, s - 1\}$ . The desired HSASs filled with pairs of points from  $V$  and indexed by  $S$  are presented as follows.

$s = 19$ :

(2, 13; 3, 17)	(3, 7; 0, 16)	(14, 8; 10, 14)	(13, 16; 11, 9)	(9, 1; 17, 7)	(14, 7; 12, 17)	(0, 15; 0, 2)	(15, 6; 4, 1)	(9, 6; 14, 18)
(13, 0; 18, 5)	(4, 12; 5, 2)	(14, 16; 15, 3)	(8, 3; 9, 13)	(5, 1; 2, 8)	(16, 10; 13, 10)	(15, 12; 11, 3)	(9, 11; 16, 3)	(15, 11; 8, 17)
(17, 7; 2, 10)	(17, 13; 8, 6)	(12, 5; 7, 18)	(8, 7; 3, 18)	(7, 10; 11, 7)	(13, 14; 0, 1)	(8, 6; 5, 16)	(4, 1; 3, 4)	(11, 12; 0, 9)
(5, 13; 10, 16)	(18, 1; 12, 15)	(18, 5; 3, 0)	(0, 2; 4, 14)	(2, 6; 2, 13)	(6, 18; 10, 11)	(11, 1; 5, 10)	(3, 5; 14, 11)	(14, 18; 13, 6)
(13, 10; 14, 15)	(6, 4; 17, 0)	(14, 2; 9, 8)	(10, 8; 0, 8)	(4, 11; 7, 6)	(17, 4; 1, 9)	(12, 2; 1, 10)	(14, 12; 16, 4)	(0, 4; 13, 8)
(10, 2; 6, 16)	(10, 3; 3, 5)	(7, 5; 9, 4)	(9, 8; 1, 11)	(18, 8; 7, 4)	(14, 3; 18, 2)	(7, 1; 13, 1)	(0, 9; 6, 9)	(1, 0; 16, 11)
(17, 12; 14, 13)	(15, 1; 9, 18)	(5, 9; 13, 12)	(10, 18; 2, 9)	(14, 17; 5, 11)	(3, 12; 17, 6)	(8, 15; 12, 6)	(15, 4; 14, 16)	(5, 10; 17, 1)
(9, 18; 5, 8)	(13, 15; 13, 7)	(3, 0; 10, 12)	(17, 3; 15, 4)	(12, 16; 8, 12)	(4, 2; 12, 18)	(1, 16; 6, 0)	(17, 6; 12, 3)	(11, 18; 1, 14)
(16, 0; 7, 1)	(9, 4; 15, 10)	(10, 11; 18, 4)	(8, 0; 17, 15)	(5, 15; 5, 15)	(16, 7; 5, 14)	(2, 11; 15, 11)	(16, 9; 2, 4)	(17, 2; 0, 7)
(13, 11; 2, 12)	(7, 6; 15, 6)	(3, 6; 7, 8)						

$s = 21$ :

(3, 16; 12, 5)	(16, 8; 17, 0)	(7, 17; 9, 0)	(2, 3; 15, 17)	(11, 3; 4, 19)	(13, 0; 8, 20)	(13, 6; 17, 12)	(12, 8; 6, 19)	(14, 12; 20, 17)
(13, 11; 9, 5)	(16, 14; 8, 9)	(8, 7; 16, 4)	(18, 2; 0, 16)	(15, 5; 16, 19)	(9, 4; 1, 3)	(15, 16; 7, 4)	(3, 17; 7, 8)	(10, 5; 18, 3)
(2, 1; 7, 20)	(1, 15; 3, 15)	(4, 16; 16, 14)	(6, 2; 3, 6)	(10, 14; 0, 15)	(5, 11; 11, 20)	(18, 13; 15, 4)	(4, 10; 13, 20)	(16, 2; 1, 2)
(6, 10; 16, 7)	(4, 0; 7, 18)	(15, 11; 2, 6)	(6, 3; 14, 10)	(16, 7; 3, 10)	(15, 6; 20, 1)	(11, 4; 10, 15)	(14, 8; 7, 3)	(11, 17; 16, 1)
(1, 16; 11, 13)	(5, 2; 10, 8)	(7, 4; 17, 11)	(2, 4; 19, 12)	(17, 9; 6, 12)	(14, 6; 5, 19)	(3, 8; 20, 2)	(18, 9; 14, 2)	(10, 11; 8, 17)
(15, 2; 11, 9)	(8, 18; 11, 8)	(7, 14; 14, 6)	(3, 0; 16, 3)	(3, 9; 11, 0)	(4, 15; 5, 8)	(13, 17; 3, 11)	(12, 4; 2, 9)	(18, 10; 5, 10)
(12, 6; 4, 11)	(17, 8; 10, 13)	(2, 12; 13, 18)	(3, 13; 6, 13)	(7, 1; 19, 8)	(5, 18; 17, 6)	(1, 0; 0, 14)	(12, 5; 5, 7)	(13, 12; 1, 0)
(12, 9; 8, 15)	(18, 1; 9, 12)	(7, 9; 20, 5)	(11, 12; 3, 14)	(11, 8; 18, 12)	(0, 17; 17, 2)	(12, 1; 16, 10)	(1, 10; 1, 6)	(6, 1; 2, 18)
(0, 16; 6, 15)	(5, 17; 14, 15)	(0, 15; 12, 13)	(7, 10; 12, 2)	(9, 13; 7, 19)	(0, 8; 5, 1)	(0, 10; 9, 19)	(5, 4; 4, 0)	(1, 17; 4, 5)
(2, 10; 4, 14)	(18, 7; 7, 13)	(9, 15; 10, 17)	(6, 11; 13, 0)	(13, 15; 18, 14)	(6, 8; 15, 9)	(3, 7; 1, 18)	(9, 14; 4, 18)	(14, 0; 10, 11)
(14, 13; 2, 16)	(5, 9; 9, 13)	(5, 14; 1, 12)						

$s = 23$ :

(6, 7; 15, 7)	(2, 5; 12, 20)	(13, 1; 14, 4)	(2, 18; 14, 15)	(14, 9; 12, 21)	(8, 10; 15, 5)	(1, 17; 7, 12)	(4, 13; 11, 10)	(4, 10; 9, 19)
(16, 9; 4, 17)	(7, 9; 11, 2)	(3, 4; 4, 20)	(6, 18; 8, 13)	(0, 16; 12, 9)	(17, 2; 6, 16)	(0, 3; 19, 2)	(15, 0; 5, 16)	(5, 15; 0, 3)
(5, 17; 14, 17)	(5, 3; 21, 16)	(1, 12; 13, 22)	(13, 14; 15, 22)	(10, 15; 10, 8)	(11, 15; 21, 7)	(6, 1; 10, 0)	(14, 4; 16, 18)	(3, 2; 8, 17)
(15, 1; 6, 1)	(12, 0; 11, 18)	(3, 0; 7, 10)	(8, 12; 21, 2)	(9, 17; 5, 18)	(14, 10; 7, 20)	(10, 11; 1, 17)	(18, 13; 9, 2)	(16, 12; 0, 15)
(9, 8; 20, 16)	(6, 8; 9, 11)	(3, 14; 9, 5)	(10, 0; 22, 14)	(3, 13; 12, 18)	(1, 4; 21, 5)	(16, 5; 2, 1)	(0, 9; 13, 15)	(5, 11; 9, 15)
(2, 0; 4, 21)	(14, 11; 13, 4)	(7, 4; 22, 3)	(7, 1; 8, 19)	(15, 17; 19, 13)	(2, 15; 11, 22)	(2, 6; 2, 5)	(8, 13; 19, 1)	(15, 4; 15, 2)
(14, 16; 6, 3)	(0, 1; 17, 20)	(6, 13; 17, 21)	(6, 9; 19, 3)	(11, 16; 10, 14)	(10, 3; 13, 11)	(12, 9; 6, 7)	(15, 7; 17, 9)	(16, 2; 18, 19)
(7, 13; 6, 20)	(10, 17; 0, 2)	(18, 0; 10, 3)	(13, 11; 3, 16)	(17, 7; 1, 4)	(5, 6; 6, 4)	(15, 6; 20, 14)	(14, 7; 14, 0)	(9, 5; 10, 22)
(2, 9; 0, 9)	(16, 1; 11, 16)	(13, 16; 5, 8)	(12, 10; 3, 4)	(0, 4; 8, 6)	(13, 0; 7, 0)	(7, 8; 10, 12)	(14, 2; 1, 10)	(5, 8; 18, 8)
(4, 16; 13, 7)	(5, 7; 5, 13)	(17, 14; 8, 11)	(12, 14; 19, 17)	(7, 10; 18, 21)	(8, 3; 22, 7)	(15, 18; 4, 18)	(17, 3; 15, 3)	(11, 6; 22, 12)
(5, 18; 19, 7)	(9, 3; 1, 14)	(1, 11; 18, 2)	(3, 11; 6, 0)	(4, 12; 12, 14)	(11, 18; 11, 5)	(2, 8; 3, 13)	(12, 11; 8, 20)	(12, 6; 16, 1)
(18, 10; 16, 12)	(18, 8; 6, 17)	(4, 18; 0, 1)						



$s = 25$ :

(8, 16; 17, 18)	(0, 3; 0, 22)	(10, 11; 4, 14)	(12, 13; 23, 20)	(5, 6; 2, 23)	(9, 10; 13, 16)	(1, 17; 5, 18)	(15, 4; 17, 12)	(5, 0; 15, 20)
(7, 8; 12, 0)	(16, 3; 15, 13)	(18, 1; 13, 12)	(18, 13; 15, 17)	(9, 0; 12, 1)	(6, 7; 4, 16)	(14, 11; 23, 13)	(6, 16; 8, 5)	(17, 11; 8, 21)
(12, 2; 13, 11)	(5, 18; 10, 18)	(12, 14; 4, 17)	(15, 6; 22, 13)	(14, 10; 15, 1)	(11, 1; 17, 0)	(10, 0; 5, 9)	(7, 4; 5, 20)	(2, 10; 2, 22)
(11, 6; 20, 7)	(17, 15; 0, 15)	(1, 0; 3, 21)	(5, 9; 4, 3)	(12, 1; 2, 19)	(15, 13; 3, 18)	(4, 2; 15, 21)	(8, 17; 10, 16)	(7, 14; 21, 22)
(14, 9; 5, 10)	(14, 0; 18, 8)	(7, 1; 23, 14)	(11, 12; 15, 5)	(13, 4; 8, 13)	(3, 7; 8, 2)	(3, 5; 5, 17)	(0, 13; 4, 24)	(3, 13; 16, 1)
(5, 2; 8, 0)	(11, 3; 9, 19)	(17, 13; 2, 6)	(15, 1; 20, 1)	(6, 2; 24, 18)	(18, 12; 6, 0)	(9, 16; 20, 0)	(16, 15; 21, 16)	(17, 2; 7, 17)
(15, 9; 19, 23)	(16, 1; 4, 7)	(18, 3; 3, 11)	(8, 0; 19, 13)	(17, 5; 1, 13)	(12, 10; 24, 8)	(15, 18; 5, 14)	(9, 8; 2, 24)	(5, 4; 9, 14)
(18, 8; 21, 20)	(16, 7; 6, 19)	(14, 8; 7, 9)	(2, 14; 3, 20)	(7, 11; 24, 1)	(11, 0; 10, 2)	(9, 3; 21, 6)	(2, 11; 6, 12)	(8, 15; 8, 4)
(16, 13; 9, 12)	(11, 5; 16, 11)	(14, 4; 19, 24)	(4, 18; 7, 1)	(12, 6; 21, 12)	(15, 12; 10, 9)	(13, 5; 19, 22)	(6, 4; 11, 0)	(10, 8; 23, 11)
(0, 6; 6, 17)	(1, 4; 10, 22)	(2, 3; 23, 10)	(14, 15; 11, 6)	(0, 4; 16, 23)	(17, 0; 11, 14)	(3, 14; 12, 14)	(13, 7; 10, 11)	(18, 14; 2, 16)
(12, 8; 3, 22)	(10, 4; 6, 18)	(9, 13; 14, 7)	(9, 11; 18, 22)	(6, 10; 10, 19)	(10, 13; 21, 0)	(17, 3; 4, 20)	(9, 1; 11, 15)	(17, 6; 9, 3)
(7, 10; 3, 17)	(8, 6; 14, 15)	(9, 18; 9, 8)	(16, 4; 3, 2)	(15, 3; 7, 24)	(12, 7; 18, 7)	(1, 2; 9, 16)	(5, 10; 7, 12)	(18, 2; 4, 19)
(5, 1; 24, 6)	(8, 2; 1, 5)	(12, 16; 1, 14)						

$s = 27$ :

(8, 5; 17, 23)	(16, 9; 11, 7)	(6, 15; 0, 10)	(1, 3; 11, 25)	(1, 10; 1, 22)	(16, 14; 13, 22)	(2, 10; 5, 0)	(4, 11; 18, 2)	(4, 10; 9, 23)
(8, 6; 9, 2)	(11, 0; 24, 15)	(0, 8; 26, 18)	(5, 16; 20, 12)	(3, 13; 26, 12)	(5, 11; 8, 7)	(10, 17; 14, 19)	(7, 15; 16, 18)	(3, 8; 20, 21)
(15, 0; 12, 8)	(6, 9; 12, 13)	(13, 12; 4, 16)	(8, 10; 10, 24)	(2, 16; 17, 2)	(18, 3; 0, 23)	(10, 3; 6, 7)	(15, 2; 6, 21)	(14, 17; 10, 2)
(4, 16; 16, 10)	(17, 7; 1, 12)	(17, 13; 17, 18)	(2, 6; 22, 26)	(9, 13; 21, 1)	(6, 5; 6, 1)	(16, 10; 18, 15)	(12, 0; 9, 1)	(17, 2; 4, 23)
(18, 4; 14, 3)	(7, 18; 13, 5)	(11, 15; 25, 19)	(18, 10; 4, 2)	(5, 14; 16, 0)	(13, 6; 5, 3)	(17, 0; 21, 16)	(3, 16; 14, 4)	(17, 4; 8, 6)
(1, 8; 12, 19)	(5, 7; 10, 11)	(16, 7; 0, 9)	(15, 3; 5, 9)	(5, 9; 14, 5)	(18, 0; 17, 19)	(15, 8; 1, 14)	(8, 13; 6, 13)	(4, 9; 22, 25)
(15, 12; 2, 22)	(5, 13; 19, 9)	(12, 11; 14, 21)	(4, 6; 4, 11)	(0, 14; 14, 6)	(13, 2; 15, 25)	(12, 14; 20, 23)	(1, 2; 16, 7)	(9, 11; 3, 0)
(2, 4; 24, 19)	(15, 10; 17, 13)	(7, 12; 7, 24)	(6, 18; 21, 7)	(7, 13; 14, 2)	(9, 2; 18, 10)	(4, 7; 26, 20)	(7, 14; 8, 19)	(7, 3; 15, 3)
(2, 14; 12, 3)	(11, 1; 20, 4)	(18, 8; 11, 16)	(16, 8; 3, 8)	(14, 15; 4, 7)	(13, 0; 7, 0)	(6, 14; 25, 18)	(11, 2; 11, 1)	(11, 16; 6, 5)
(11, 18; 12, 10)	(9, 3; 16, 2)	(17, 1; 0, 13)	(15, 9; 23, 24)	(1, 13; 8, 23)	(1, 12; 10, 17)	(5, 10; 26, 3)	(9, 8; 4, 15)	(5, 17; 15, 22)
(17, 15; 11, 3)	(12, 4; 5, 15)	(18, 13; 20, 22)	(1, 15; 26, 15)	(5, 3; 24, 18)	(12, 10; 12, 11)	(6, 10; 20, 8)	(17, 8; 7, 5)	(13, 14; 24, 11)
(14, 11; 26, 9)	(16, 6; 19, 23)	(17, 9; 20, 9)	(0, 3; 10, 22)	(18, 12; 18, 6)	(11, 7; 22, 23)	(1, 0; 3, 2)	(12, 3; 8, 13)	(18, 2; 8, 9)
(14, 1; 21, 5)	(12, 8; 0, 25)	(1, 6; 14, 24)	(0, 5; 25, 4)	(2, 0; 20, 13)	(11, 6; 17, 16)	(9, 12; 19, 26)	(4, 5; 13, 21)	(9, 7; 17, 6)
(7, 10; 21, 25)	(14, 18; 1, 15)	(3, 4; 1, 17)						

$s = 29$ :

(14, 3; 12, 3)	(2, 4; 23, 20)	(17, 12; 18, 16)	(12, 4; 28, 11)	(1, 11; 14, 24)	(3, 10; 8, 26)	(14, 10; 21, 22)	(18, 8; 3, 20)	(5, 15; 26, 11)
(2, 6; 2, 4)	(8, 1; 5, 17)	(12, 11; 4, 8)	(8, 5; 4, 14)	(4, 9; 19, 26)	(11, 4; 21, 18)	(9, 15; 6, 18)	(8, 11; 23, 27)	(7, 14; 10, 6)
(8, 15; 21, 7)	(9, 16; 10, 17)	(16, 15; 9, 15)	(1, 6; 10, 27)	(15, 2; 12, 0)	(5, 6; 19, 22)	(7, 0; 16, 0)	(15, 10; 23, 1)	(0, 17; 21, 12)
(16, 14; 4, 18)	(13, 7; 24, 26)	(15, 1; 16, 28)	(12, 6; 3, 17)	(9, 13; 21, 14)	(6, 17; 24, 0)	(15, 7; 27, 5)	(11, 2; 3, 10)	(10, 11; 25, 20)
(7, 12; 2, 14)	(16, 7; 19, 23)	(18, 14; 17, 9)	(4, 8; 22, 12)	(7, 9; 13, 11)	(6, 11; 16, 26)	(12, 18; 21, 19)	(3, 9; 27, 24)	(7, 1; 21, 1)
(17, 11; 9, 22)	(3, 4; 2, 7)	(4, 18; 1, 4)	(6, 3; 5, 21)	(2, 8; 26, 13)	(15, 18; 8, 2)	(17, 4; 8, 6)	(16, 10; 7, 6)	(10, 17; 14, 5)
(14, 0; 27, 14)	(18, 5; 7, 24)	(0, 18; 11, 10)	(0, 10; 24, 18)	(0, 13; 20, 8)	(13, 8; 19, 18)	(15, 3; 14, 20)	(7, 18; 18, 22)	(6, 7; 12, 28)
(8, 9; 8, 16)	(16, 12; 13, 20)	(3, 11; 15, 1)	(0, 15; 4, 3)	(9, 12; 15, 0)	(3, 5; 9, 18)	(9, 5; 28, 25)	(5, 12; 1, 10)	(6, 18; 15, 25)
(1, 13; 11, 12)	(3, 2; 17, 28)	(10, 4; 16, 10)	(9, 17; 23, 3)	(14, 1; 26, 23)	(15, 2; 27, 21)	(13, 2; 7, 9)	(13, 10; 28, 0)	(3, 18; 6, 23)
(8, 3; 11, 0)	(6, 0; 7, 1)	(0, 16; 5, 25)	(13, 17; 1, 17)	(6, 16; 14, 11)	(12, 2; 24, 25)	(16, 8; 24, 1)	(10, 8; 15, 2)	(14, 2; 8, 1)
(0, 11; 17, 19)	(13, 15; 10, 22)	(8, 0; 28, 6)	(13, 6; 13, 6)	(1, 4; 3, 25)	(14, 11; 28, 7)	(17, 14; 2, 11)	(13, 12; 5, 23)	(16, 5; 12, 16)
(9, 1; 4, 7)	(10, 12; 27, 12)	(5, 7; 8, 3)	(10, 1; 9, 13)	(9, 2; 22, 5)	(16, 13; 2, 3)	(13, 3; 25, 16)	(9, 11; 2, 12)	(3, 17; 4, 10)
(11, 18; 0, 13)	(2, 10; 11, 19)	(17, 7; 7, 15)	(18, 2; 16, 14)	(17, 1; 20, 19)	(17, 15; 13, 25)	(12, 0; 26, 9)	(7, 10; 4, 17)	(5, 14; 15, 20)
(13, 4; 27, 15)	(5, 11; 5, 6)	(1, 2; 18, 15)	(8, 7; 9, 25)	(9, 6; 9, 20)	(4, 5; 13, 17)	(3, 0; 22, 13)	(15, 14; 24, 19)	(4, 14; 0, 5)
(5, 0; 2, 23)	(12, 1; 6, 22)	(1, 16; 0, 8)						

$s = 31$ :

(0, 6; 25, 1)	(3, 9; 3, 15)	(0, 9; 12, 5)	(8, 18; 26, 22)	(11, 8; 19, 1)	(11, 7; 10, 30)	(18, 12; 21, 27)	(3, 12; 11, 17)	(13, 3; 1, 12)
(4, 3; 21, 6)	(10, 1; 26, 13)	(10, 18; 11, 15)	(14, 15; 21, 5)	(11, 3; 20, 28)	(18, 4; 2, 16)	(17, 8; 27, 23)	(5, 0; 21, 20)	(12, 10; 9, 10)
(7, 5; 26, 6)	(0, 10; 14, 18)	(0, 12; 24, 7)	(16, 15; 4, 16)	(3, 5; 16, 27)	(15, 10; 8, 1)	(17, 13; 15, 0)	(3, 10; 30, 0)	(9, 8; 16, 13)
(18, 11; 5, 8)	(9, 14; 25, 9)	(9, 12; 29, 0)	(6, 2; 12, 15)	(16, 12; 3, 8)	(2, 16; 7, 23)	(13, 7; 29, 25)	(6, 12; 14, 6)	(15, 2; 29, 17)
(6, 5; 24, 28)	(8, 7; 24, 5)	(18, 13; 4, 23)	(5, 16; 0, 25)	(0, 15; 0, 6)	(2, 4; 11, 27)	(0, 1; 28, 8)	(10, 16; 6, 24)	(6, 16; 11, 26)
(16, 4; 5, 9)	(6, 17; 18, 21)	(3, 14; 8, 23)	(11, 5; 4, 12)	(11, 15; 22, 9)	(14, 2; 2, 24)	(5, 4; 13, 1)	(8, 1; 21, 0)	(17, 12; 1, 16)
(1, 4; 4, 29)	(15, 9; 24, 11)	(4, 8; 25, 18)	(14, 13; 28, 22)	(1, 11; 17, 7)	(3, 6; 5, 13)	(14, 6; 4, 27)	(9, 5; 23, 18)	(17, 10; 4, 20)
(14, 5; 15, 29)	(18, 1; 12, 6)	(17, 14; 11, 6)	(1, 2; 22, 25)	(13, 15; 27, 10)	(2, 11; 16, 0)	(16, 14; 13, 17)	(2, 18; 10, 14)	(3, 7; 18, 9)
(1, 17; 9, 14)	(11, 17; 2, 25)	(18, 15; 3, 18)	(16, 11; 15, 18)	(14, 0; 19, 16)	(17, 0; 22, 10)	(15, 1; 23, 15)	(7, 2; 21, 8)	(13, 10; 16, 17)
(2, 9; 28, 6)	(3, 1; 24, 19)	(9, 1; 20, 30)	(17, 9; 26, 7)	(8, 16; 20, 12)	(0, 4; 17, 23)	(9, 18; 17, 1)	(6, 1; 16, 10)	(16, 3; 22, 2)
(9, 16; 19, 10)	(0, 11; 29, 27)	(15, 12; 25, 12)	(14, 8; 30, 3)	(7, 14; 7, 20)	(7, 1; 11, 3)	(16, 13; 14, 21)	(8, 15; 28, 2)	(6, 13; 30, 7)
(10, 7; 2, 12)	(7, 9; 14, 4)	(5, 13; 9, 11)	(4, 7; 19, 0)	(12, 5; 19, 22)	(0, 2; 30, 9)	(6, 8; 17, 9)	(12, 2; 20, 13)	(6, 4; 3, 22)
(12, 13; 26, 5)	(7, 12; 28, 23)	(10, 2; 3, 5)	(7, 0; 13, 15)	(12, 4; 30, 15)	(8, 13; 8, 6)	(10, 9; 22, 27)	(17, 4; 24, 8)	(3, 2; 4, 26)
(13, 11; 24, 13)	(10, 11; 23, 21)	(9, 6; 8, 2)	(14, 11; 26, 14)	(18, 6; 20, 0)	(2, 13; 19, 18)	(8, 0; 11, 4)	(3, 18; 25, 7)	(17, 5; 3, 17)
(4, 14; 10, 12)	(4, 10; 28, 7)	(7, 16; 1, 27)	(1, 5; 5, 2)	(14, 1; 18, 1)	(0, 13; 3, 2)	(15, 5; 14, 30)	(15, 17; 19, 13)	(6, 10; 19, 29)
(3, 8; 14, 29)	(4, 15; 20, 26)	(5, 8; 7, 10)						

$s = 33$ :

(2, 4; 22, 0)	(17, 11; 23, 7)	(5, 15; 9, 28)	(6, 14; 10, 23)	(9, 15; 22, 8)	(12, 5; 26, 29)	(17, 2; 1, 13)	(2, 13; 16, 31)	(13, 6; 0, 28)
(4, 16; 27, 5)	(10, 2; 14, 7)	(9, 11; 26, 21)	(16, 1; 22, 10)	(11, 1; 0, 17)	(14, 12; 18, 1)	(17, 13; 3, 19)	(2, 7; 25, 24)	(7, 17; 11, 22)
(5, 1; 11, 25)	(6, 1; 29, 13)	(17, 1; 21, 14)	(6, 12; 32, 3)	(14, 16; 15, 8)	(9, 3; 18, 24)	(1, 14; 19, 5)	(13, 15; 18, 21)	(0, 3; 21, 3)
(12, 7; 0, 12)	(5, 8; 32, 0)	(11, 0; 16, 18)	(16, 13; 20, 17)	(1, 2; 28, 6)	(9, 4; 1, 10)	(2, 5; 20, 27)	(1, 13; 32, 8)	(11, 18; 27, 3)
(5, 4; 31, 15)	(17, 6; 16, 5)	(12, 1; 24, 4)	(9, 16; 23, 0)	(8, 13; 6, 27)	(1, 4; 3, 23)	(15, 4; 6, 30)	(4, 11; 19, 2)	(10, 6; 25, 21)
(16, 11; 13, 6)	(0, 10; 0, 6)	(3, 8; 13, 5)	(6, 3; 31, 6)	(10, 15; 1, 27)	(18, 2; 18, 5)	(3, 18; 12, 25)	(18, 7; 15, 1)	(12, 13; 5, 7)
(5, 10; 3, 30)	(10, 11; 20, 10)	(5, 16; 21, 4)	(14, 13; 2, 14)	(11, 3; 32, 1)	(10, 14; 32, 12)	(0, 16; 24, 1)	(10, 9; 28, 5)	(18, 8; 21, 10)
(7, 4; 28, 32)	(6, 0; 11, 9)	(15, 0; 32, 2)	(8, 7; 19, 29)	(3, 13; 10, 11)	(10, 13; 24, 9)	(14, 5; 24, 7)	(6, 7; 2, 30)	(7, 9; 9, 3)
(0, 18; 17, 23)	(0, 2; 10, 19)	(11, 14; 28, 30)	(18, 4; 29, 24)	(14, 8; 31, 3)	(15, 6; 7, 12)	(14, 18; 9, 26)	(5, 13; 1, 22)	(17, 0; 8, 28)
(7, 13; 4, 26)	(15, 17; 20, 24)	(5, 3; 23, 8)	(9, 2; 11, 32)	(6, 2; 17, 26)	(9, 1; 12, 30)	(6, 18; 4, 19)	(4, 6; 8, 20)	(15, 7; 10, 13)
(13, 4; 25, 13)	(11, 8; 25, 15)	(10, 3; 17, 4)	(15, 2; 29, 15)	(0, 5; 5, 12)	(7, 11; 14, 8)	(7, 1; 27, 31)	(0, 9; 31, 29)	(18, 5; 16, 2)
(16, 8; 28, 14)	(12, 9; 13, 16)	(17, 9; 25, 4)	(18, 15; 11, 14)	(5, 6; 18, 14)	(9, 5; 6, 19)	(14, 4; 11, 16)	(7, 14; 21, 6)	(0, 8; 7, 22)
(10, 1; 18, 2)	(17, 5; 10, 17)	(1, 3; 15, 9)	(7, 10; 16, 23)	(1, 0; 26, 20)	(3, 12; 14, 30)	(13, 11; 29, 12)	(12, 4; 21, 17)	(12, 8; 11, 20)
(17, 12; 15, 6)	(10, 16; 11, 29)	(18, 10; 8, 13)	(15, 14; 25, 17)	(0, 4; 4, 14)	(9, 18; 7, 20)	(17, 4; 18, 9)	(16, 7; 7, 18)	(3, 17; 2, 27)
(12, 10; 31, 19)	(3, 4; 26, 7)	(11, 6; 22, 24)	(12, 18; 22, 28)	(8, 15; 4, 23)	(11, 15; 31, 5)	(3, 15; 19, 0)	(9, 8; 2, 17)	(1, 8; 16, 1)
(16, 2; 12, 3)	(16, 15; 26, 16)	(2, 11; 9, 4)	(9, 6; 27, 15)	(14, 17; 0, 29)	(14, 3; 20, 22)	(2, 8; 30, 8)	(13, 0; 15, 30)	(0, 14; 13, 27)
(17, 8; 26, 12)	(12, 16; 25, 9)	(2, 12; 2, 23)						

$s = 35$ :

(9, 4; 22, 15)	(0, 2; 14, 34)	(13, 18; 25, 8)	(11, 16; 4, 30)	(6, 18; 3, 17)	(13, 15; 16, 26)	(2, 6; 27, 23)	(3, 1; 19, 15)	(3, 13; 22, 7)
(6, 1; 6, 31)	(5, 17; 16, 5)	(14, 1; 26, 32)	(9, 6; 12, 34)	(1, 17; 3, 22)	(5, 3; 9, 31)	(11, 3; 20, 1)	(11, 15; 23, 12)	(17, 12; 0, 20)
(4, 6; 16, 9)	(16, 1; 9, 29)	(12, 14; 15, 27)	(6, 8; 30, 15)	(3, 7; 6, 34)	(11, 4; 7, 10)	(15, 17; 27, 9)	(10, 8; 5, 27)	(7, 8; 8, 22)
(16, 15; 8, 15)	(9, 12; 31, 16)	(9, 13; 2, 14)	(12, 18; 21, 12)	(8, 2; 31, 32)	(12, 6; 8, 10)	(1, 13; 5, 34)	(12, 5; 28, 34)	(13, 5; 6, 13)
(5, 10; 17, 15)	(12, 2; 4, 26)	(10, 2; 22, 18)	(6, 11; 28, 32)	(4, 17; 4, 12)	(16, 0; 16, 2)	(6, 7; 1, 26)	(7, 2; 10, 2)	(18, 9; 28, 0)
(13, 11; 18, 17)	(10, 15; 34, 2)	(16, 3; 26, 27)	(2, 1; 12, 17)	(9, 11; 27, 8)	(6, 14; 25, 18)	(9, 17; 21, 17)	(18, 11; 9, 26)	(10, 17; 1, 30)
(8, 0; 6, 12)	(14, 4; 34, 21)	(7, 15; 24, 32)	(14, 9; 33, 19)	(17, 14; 28, 8)	(12, 11; 11, 13)	(10, 4; 13, 0)	(4, 2; 25, 33)	(2, 3; 21, 0)
(18, 0; 18, 15)	(10, 13; 12, 33)	(16, 2; 20, 24)	(14, 11; 2, 31)	(3, 12; 25, 17)	(1, 15; 11, 25)	(1, 12; 2, 30)	(12, 4; 19, 6)	(9, 7; 25, 7)
(3, 9; 3, 29)	(2, 13; 1, 28)	(18, 15; 6, 30)	(15, 4; 31, 5)	(16, 13; 19, 0)	(12, 8; 33, 18)	(4, 18; 1, 24)	(3, 17; 10, 18)	(6, 10; 21, 24)
(5, 2; 30, 8)	(10, 14; 16, 23)	(9, 8; 9, 20)	(2, 17; 7, 19)	(2, 18; 11, 16)	(15, 8; 1, 19)	(11, 2; 5, 15)	(3, 8; 23, 28)	(5, 6; 0, 11)
(15, 14; 10, 0)	(0, 3; 30, 33)	(5, 11; 19, 3)	(6, 17; 2, 13)	(14, 16; 22, 11)	(1, 18; 23, 20)	(0, 4; 3, 26)	(9, 1; 13, 1)	(12, 15; 29, 7)
(3, 10; 8, 32)	(0, 5; 10, 1)	(11, 0; 21, 22)	(0, 10; 19, 4)	(13, 12; 32, 3)	(13, 14; 9, 30)	(17, 13; 31, 15)	(16, 10; 25, 28)	(11, 8; 34, 0)
(7, 18; 19, 5)	(17, 8; 14, 11)	(5, 18; 22, 27)	(8, 14; 4, 17)	(18, 8; 2, 29)	(14, 7; 3, 20)	(12, 16; 1, 14)	(5, 1; 33, 21)	(15, 2; 13, 3)
(11, 17; 6, 24)	(7, 10; 11, 9)	(0, 9; 11, 32)	(13, 7; 21, 23)	(3, 14; 12, 24)	(0, 12; 9, 23)	(7, 1; 4, 28)	(14, 2; 6, 29)	(9, 5; 24, 23)
(5, 4; 2, 32)	(8, 16; 21, 3)	(1, 0; 24, 8)	(18, 10; 10, 31)	(15, 9; 4, 18)	(17, 0; 29, 25)	(10, 9; 6, 26)	(15, 6; 22, 33)	(0, 14; 13, 5)
(7, 0; 27, 0)	(9, 16; 5, 10)	(13, 4; 20, 11)	(3, 18; 4, 13)	(14, 18; 7, 14)	(4, 16; 23, 17)	(5, 15; 14, 20)	(7, 4; 18, 30)	(10, 11; 29, 14)
(6, 0; 7, 20)	(0, 15; 28, 17)	(13, 6; 29, 4)	(11, 7; 33, 16)	(6, 3; 14, 5)	(5, 16; 7, 18)	(13, 8; 24, 10)	(1, 4; 27, 14)	(8, 1; 16, 7)
(16, 7; 31, 13)	(5, 7; 12, 29)	(5, 8; 25, 26)						

■

**Lemma 8.16:** There exists an HSAS( $s, v; 5, 3$ ) for each  $(s, v) \in \{(21, 11), (29, 15), (37, 19)\}$ .

*Proof:* Let  $V = I_v$  and  $S = I_s$ . Let  $W = \{v - 3, v - 2, v - 1\}$  and  $T = \{s - 5, s - 4, s - 3, s - 2, s - 1\}$ . The desired HSASs filled with pairs of points from  $V$  and indexed by  $S$  are presented as follows.

$(s, v) = (21, 11)$ :

(1, 10; 4, 12)	(3, 6; 20, 6)	(4, 8; 12, 3)	(2, 0; 1, 18)	(7, 5; 5, 17)	(8, 7; 13, 8)	(3, 1; 19, 0)	(6, 1; 11, 5)	(9, 7; 11, 1)	(0, 6; 17, 0)
(2, 6; 3, 9)	(2, 3; 17, 15)	(10, 7; 6, 9)	(7, 1; 15, 18)	(0, 10; 13, 2)	(1, 5; 7, 9)	(6, 10; 1, 10)	(4, 7; 0, 20)	(10, 4; 14, 11)	(5, 0; 8, 12)
(5, 8; 11, 10)	(0, 3; 16, 11)	(4, 2; 10, 19)	(2, 9; 4, 7)	(4, 5; 6, 16)	(7, 6; 16, 7)	(0, 7; 3, 19)	(3, 5; 18, 14)	(4, 9; 2, 9)	(4, 0; 7, 5)
(2, 8; 6, 0)	(3, 10; 7, 3)	(2, 1; 13, 16)	(9, 3; 5, 13)	(4, 6; 13, 18)	(3, 4; 4, 8)	(8, 0; 15, 4)	(1, 4; 17, 1)	(8, 6; 2, 14)	(10, 2; 5, 8)
(0, 9; 6, 10)	(1, 0; 14, 20)	(5, 2; 2, 20)	(7, 3; 2, 10)	(5, 6; 19, 4)	(3, 8; 1, 9)	(5, 10; 15, 0)	(7, 2; 14, 12)	(9, 6; 12, 15)	(1, 9; 3, 8)

$(s, v) = (29, 15)$ :

(3, 0; 16, 25)	(8, 4; 28, 23)	(4, 9; 24, 8)	(13, 8; 0, 15)	(12, 5; 8, 2)	(11, 5; 4, 28)	(10, 5; 14, 21)	(2, 7; 22, 11)	(4, 12; 3, 15)
(6, 7; 5, 19)	(12, 10; 18, 6)	(3, 1; 6, 24)	(3, 11; 27, 5)	(6, 1; 28, 2)	(9, 13; 16, 5)	(14, 7; 18, 4)	(11, 0; 24, 22)	(12, 6; 22, 7)
(12, 7; 16, 14)	(10, 4; 13, 2)	(10, 6; 23, 17)	(10, 9; 28, 3)	(10, 0; 27, 19)	(14, 10; 22, 8)	(3, 6; 21, 13)	(3, 12; 20, 0)	(7, 0; 28, 0)
(13, 6; 1, 8)	(5, 8; 22, 6)	(4, 1; 1, 11)	(7, 11; 17, 3)	(12, 1; 21, 4)	(2, 14; 3, 21)	(13, 4; 14, 4)	(11, 9; 26, 14)	(13, 2; 23, 2)
(14, 0; 15, 2)	(14, 9; 19, 1)	(10, 7; 20, 26)	(8, 1; 10, 26)	(2, 0; 10, 6)	(1, 7; 9, 15)	(6, 0; 11, 26)	(1, 14; 17, 5)	(14, 8; 20, 12)
(2, 12; 9, 12)	(13, 3; 10, 18)	(11, 1; 0, 16)	(0, 1; 20, 3)	(10, 3; 4, 12)	(3, 8; 11, 3)	(13, 7; 6, 21)	(6, 14; 9, 0)	(5, 6; 24, 12)
(0, 8; 7, 8)	(1, 9; 27, 7)	(4, 7; 25, 7)	(9, 2; 25, 20)	(2, 1; 14, 8)	(12, 11; 10, 1)	(9, 12; 13, 11)	(10, 11; 25, 9)	(6, 8; 14, 25)
(3, 7; 8, 23)	(11, 2; 19, 15)	(2, 4; 26, 0)	(4, 14; 16, 6)	(3, 2; 1, 28)	(6, 9; 15, 4)	(5, 7; 1, 27)	(8, 2; 4, 27)	(9, 7; 12, 10)
(5, 1; 25, 19)	(2, 6; 18, 16)	(14, 3; 7, 14)	(9, 8; 9, 18)	(14, 11; 23, 13)	(5, 3; 15, 26)	(13, 1; 22, 13)	(4, 3; 22, 19)	(13, 5; 3, 9)
(0, 4; 21, 9)	(7, 8; 24, 13)	(0, 12; 5, 23)	(13, 11; 11, 7)	(5, 0; 18, 13)	(0, 13; 12, 17)	(14, 5; 11, 10)	(10, 2; 24, 5)	(12, 8; 19, 17)
(6, 11; 20, 6)	(4, 6; 10, 27)	(11, 4; 12, 18)	(5, 9; 0, 23)	(10, 8; 16, 1)	(8, 11; 21, 2)	(3, 9; 2, 17)	(5, 4; 20, 5)	(2, 5; 7, 17)

$(s, v) = (37, 19)$ :

(5, 11; 24, 2)	(0, 7; 32, 21)	(15, 17; 1, 26)	(0, 17; 24, 4)	(5, 16; 12, 30)	(5, 1; 26, 13)	(11, 16; 1, 13)	(5, 2; 1, 22)
(11, 4; 12, 0)	(11, 7; 36, 6)	(13, 2; 8, 29)	(0, 6; 23, 25)	(6, 17; 20, 15)	(18, 9; 0, 13)	(8, 2; 16, 0)	(8, 6; 21, 31)
(15, 8; 24, 23)	(9, 15; 29, 15)	(15, 16; 11, 7)	(1, 4; 4, 25)	(11, 3; 26, 22)	(13, 1; 11, 34)	(1, 15; 9, 18)	(0, 3; 33, 0)
(7, 8; 15, 2)	(14, 12; 6, 27)	(10, 18; 15, 31)	(12, 2; 36, 3)	(6, 7; 35, 22)	(9, 12; 17, 22)	(6, 13; 10, 32)	(17, 3; 13, 2)
(8, 1; 36, 19)	(0, 5; 17, 6)	(14, 8; 35, 7)	(5, 3; 15, 21)	(4, 3; 7, 36)	(13, 0; 22, 36)	(12, 11; 28, 25)	(0, 4; 27, 8)
(13, 16; 17, 4)	(0, 12; 26, 11)	(16, 10; 16, 22)	(11, 2; 14, 9)	(7, 3; 12, 1)	(18, 8; 17, 5)	(7, 18; 8, 19)	(9, 7; 28, 20)
(15, 13; 19, 27)	(0, 10; 7, 19)	(6, 12; 24, 33)	(12, 13; 12, 9)	(18, 0; 16, 1)	(6, 16; 29, 6)	(9, 10; 11, 27)	(17, 10; 9, 17)
(6, 15; 4, 28)	(17, 1; 22, 3)	(2, 10; 4, 23)	(16, 1; 24, 14)	(4, 15; 2, 30)	(8, 11; 27, 32)	(6, 5; 7, 27)	(14, 5; 19, 9)
(5, 12; 31, 23)	(11, 10; 34, 29)	(14, 0; 13, 29)	(13, 7; 25, 30)	(17, 4; 14, 5)	(17, 2; 21, 27)	(14, 10; 8, 18)	(1, 12; 15, 32)
(7, 14; 4, 14)	(15, 0; 14, 3)	(13, 5; 16, 14)	(7, 1; 29, 0)	(6, 14; 26, 0)	(12, 15; 35, 0)	(16, 3; 18, 27)	(5, 8; 18, 4)
(3, 9; 16, 5)	(2, 18; 11, 28)	(1, 2; 6, 30)	(0, 16; 5, 15)	(1, 18; 27, 20)	(14, 18; 25, 22)	(11, 13; 7, 15)	(15, 5; 34, 10)
(6, 4; 17, 1)	(3, 1; 28, 23)	(1, 0; 10, 31)	(8, 4; 11, 22)	(0, 8; 12, 34)	(9, 5; 3, 32)	(14, 11; 30, 17)	(3, 13; 6, 35)
(11, 9; 4, 33)	(18, 4; 9, 29)	(9, 14; 34, 31)	(3, 6; 34, 30)	(4, 12; 34, 13)	(9, 4; 19, 24)	(16, 2; 31, 19)	(15, 11; 16, 20)
(1, 6; 16, 8)	(14, 15; 36, 5)	(17, 14; 12, 23)	(15, 18; 21, 6)	(14, 3; 32, 11)	(5, 10; 0, 36)	(10, 4; 6, 32)	(9, 16; 25, 2)
(9, 6; 14, 36)	(13, 17; 28, 0)	(12, 16; 21, 8)	(14, 1; 2, 33)	(4, 5; 28, 35)	(13, 9; 21, 23)	(17, 9; 6, 18)	(15, 10; 33, 25)
(10, 12; 20, 30)	(13, 10; 5, 24)	(7, 2; 34, 18)	(3, 2; 17, 25)	(5, 17; 25, 29)	(16, 4; 10, 20)	(9, 1; 12, 7)	(18, 12; 7, 18)
(2, 0; 20, 2)	(3, 12; 29, 4)	(8, 9; 26, 8)	(15, 7; 13, 17)	(13, 14; 20, 1)	(6, 11; 3, 19)	(16, 14; 28, 3)	(12, 7; 10, 5)
(1, 10; 1, 35)	(4, 14; 21, 16)	(3, 8; 20, 9)	(0, 9; 30, 9)	(6, 10; 2, 12)	(4, 13; 18, 31)	(6, 2; 5, 13)	(12, 8; 1, 14)
(11, 1; 21, 5)	(4, 2; 26, 33)	(11, 0; 35, 18)	(8, 17; 10, 30)	(18, 11; 23, 10)	(17, 7; 7, 31)	(15, 3; 8, 31)	(2, 14; 15, 24)
(7, 16; 23, 9)	(7, 5; 11, 33)	(10, 3; 14, 10)	(11, 17; 8, 11)	(7, 10; 3, 26)	(2, 9; 35, 10)	(2, 15; 32, 12)	(17, 12; 19, 16)
(18, 3; 3, 24)	(13, 18; 2, 26)	(10, 8; 13, 28)					

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